Probar que la función

$$f(x,y) := \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} & \text{si } (x,y) \neq (0,0), \\ 0 & \text{si } (x,y) = (0,0), \end{cases}$$

no es diferenciable en $(0,0)$.

Solución

Derivadas parciales:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h^3/h^2}{h} = 1$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{-k^3/k^2}{k} = -1$$

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Límite doble

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - h + k}{\sqrt{h^2+k^2}} =$$

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Solución

Límite direccional cuando $k = mh$ $(m \neq 0, m \neq 1)$

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Por tanto, no existe el límite doble; lo que implica que $f(x, y)$ no es diferenciable en $(0, 0)$.

Pero . . . ¿es $f(x, y)$ continua en $(0, 0)$?