Extensión de la regla de la cadena

Funciones diferenciables.

$$z = f(x,y), \quad x = x(u,v,w), y = y(u,v,w)$$

$$z = f\Big(x(u,v,w), y(u,v,w)\Big) \Rightarrow$$

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial w} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial w}$$

$$z'_u(u_0,v_0,w_0) = f'_x(x_0,y_0)x'_u(u_0,v_0,w_0)$$

$$+f'_y(x_0,y_0)y'_u(u_0,v_0,w_0)$$

$$x_0 := x(u_0,v_0,w_0),\ y_0 := y(u_0,v_0,w_0).$$

Ejemplo 1

$x(u,v,w) := u+v-w^2, y(u,v,w) := uv^3w$, $z=f(x,y)$. Hallar $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$ y $\frac{\partial z}{\partial w}$ en función de $\frac{\partial f}{\partial x}$ y $\frac{\partial f}{\partial y}$.

Solución

$$\frac{\partial x}{\partial u}=1, \frac{\partial x}{\partial v}=1, \frac{\partial x}{\partial w}=-2w$$

$$\frac{\partial y}{\partial u}=v^3w, \frac{\partial y}{\partial v}=3uv^2w, \frac{\partial y}{\partial w}=uv^3$$

$$\frac{\partial z}{\partial u}=\frac{\partial f}{\partial x}\,\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\,\frac{\partial y}{\partial u}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\,v^3w$$

$$\frac{\partial z}{\partial v}=\frac{\partial f}{\partial x}\,\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\,\frac{\partial y}{\partial v}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\,3uv^2w$$

$$\frac{\partial z}{\partial w}=\frac{\partial f}{\partial x}\,\frac{\partial x}{\partial w}+\frac{\partial f}{\partial y}\,\frac{\partial y}{\partial w}=-\frac{\partial f}{\partial x}\,2w+\frac{\partial f}{\partial y}\,uv^3$$

Ejemplo 2 La temperatura en una placa delgada se representa por la función real f, siendo $f(x, y)$ la temperatura en (x, y). Introduciendo coordenadas polares $x = r\cos\theta, y = r\,\mathrm{sen}\,\theta$, la temperatura se convierte en una función de r y θ determinada por la ecuación

$$\varphi(r, \theta) := f(r\cos\theta, r\,\mathrm{sen}\,\theta).$$

Expresar $\dfrac{\partial\varphi}{\partial r}$ y $\dfrac{\partial\varphi}{\partial\theta}$ en términos de $\dfrac{\partial f}{\partial x}$ y $\dfrac{\partial f}{\partial y}$.

Solución

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \mathrm{sen}\,\theta, \quad \frac{\partial x}{\partial\theta} = -r\,\mathrm{sen}\,\theta,$$

$$\frac{\partial y}{\partial\theta} = r\cos\theta$$

$$\frac{\partial\varphi}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\,\mathrm{sen}\,\theta,$$

$$\frac{\partial\varphi}{\partial\theta} = -r\frac{\partial f}{\partial x}\,\mathrm{sen}\,\theta + r\frac{\partial f}{\partial y}\cos\theta$$

Ejercicios

Ejercicio 31.-

Sean $f(x,y)$ diferenciable, $x = r\cos\theta, y = r\,\mathrm{sen}\,\theta$. Definimos $\varphi(r,\theta) = f(r\cos\theta, r\,\mathrm{sen}\,\theta)$. Hallar $\dfrac{\partial^2\varphi}{\partial\theta^2}$ en función de las derivadas parciales de $f(x,y)$.

Respuesta

$$\begin{aligned}\frac{\partial^2\varphi}{\partial\theta^2} = &-r\cos\theta\,\frac{\partial f}{\partial x} + r^2\,\mathrm{sen}^2\,\theta\,\frac{\partial^2 f}{\partial x^2}\\ &- r^2\,\mathrm{sen}\,\theta\cos\theta\,\frac{\partial^2 f}{\partial y\partial x} - r\,\mathrm{sen}\,\theta\,\frac{\partial f}{\partial y}\\ &- r^2\,\mathrm{sen}\,\theta\cos\theta\frac{\partial^2 f}{\partial x\partial y} + r^2\cos^2\theta\,\frac{\partial^2 f}{\partial y^2}.\end{aligned}$$

Ejercicio 32.-

El cambio de variables $x = u + v, y = uv^2$ transforma $f(x, y)$ en $g(u, v)$. Calcular el valor de $\dfrac{\partial^2 g}{\partial v \partial u}$ en el punto $u = 1, v = 1$, sabiendo que

$$\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 1$$

en el punto $x = 2, y = 1$.

Respuesta.- 8

Ejercicio 34.-

Sabiendo que $\nabla f(x, y, z)$ es paralelo al vector $(1, 2, -1)$ para todo (x, y, z), demostrar que $f(0, 0, -1) = f(1, -1, -2)$.