Lipschitz stability of controlled invariant subspaces. *

Juan-Miguel Gracia, Francisco E. Velasco†

Revised October 15, 2010

Abstract

Let $(A,B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ and $\mathcal{M}$ be an $(A,B)-$invariant subspace. In this paper the following results are presented: (i) If $\mathcal{M} \cap \operatorname{Im} \boldsymbol{B} = \{0\}$, necessary and sufficient conditions for the Lipschitz stability of $\mathcal{M}$ are given. (ii) If $\mathcal{M}$ contains the controllability subspace of the pair (A,B), sufficient conditions for the Lipschitz stability of the subspace $\mathcal{M}$ are given.

AMS classification: 93B10, 15A60, 15A21.
Key Words: (A,B)–invariant subspace; Lipschitz stability; Roth's criterion.

1 Introduction

Given the pair of linear maps $\boldsymbol{A}: \mathbb{C}^n \to \mathbb{C}^n$ and $\boldsymbol{B}: \mathbb{C}^m \to \mathbb{C}^n$, a subspace $\mathcal{M}$ of $\mathbb{C}^n$ is said to be $(\boldsymbol{A},\boldsymbol{B})-$invariant if $\boldsymbol{A}(\mathcal{M}) \subset \mathcal{M} + \operatorname{Im} \boldsymbol{B}$, where $\operatorname{Im} \boldsymbol{B}$ is the image subspace of $\boldsymbol{B}$.

We use the operator norm induced by the Euclidean norms on $\mathbb{C}^m$ and $\mathbb{C}^n$, and θ denotes the gap distance between subspaces. Let $\boldsymbol{A}: \mathbb{C}^n \to \mathbb{C}^n$ be a linear map. An $\boldsymbol{A}-$invariant subspace $\mathcal{N}$ of $\mathbb{C}^n$ is said to be *Lipschitz stable* if there exist $K, \varepsilon > 0$ such that every linear map $\boldsymbol{A}': \mathbb{C}^n \to \mathbb{C}^n$ that satisfies $\|\boldsymbol{A}' - \boldsymbol{A}\| < \varepsilon$ has an $\boldsymbol{A}'-$invariant subspace $\mathcal{N}'$ for which

$$\theta(\mathcal{N}', \mathcal{N}) \leq K \|\boldsymbol{A}' - \boldsymbol{A}\|.$$

A generalization of this concept was given in [4, Theorem 15.8.1, p. 468]: An $(\boldsymbol{A},\boldsymbol{B})$–invariant subspace $\mathcal{M}$ is said to be $(\boldsymbol{A},\boldsymbol{B})-$*Lipschitz stable* if there exist positive constants K and ε such that every pair of linear maps $(\boldsymbol{A}', \boldsymbol{B}')$ that satisfies

$$\|\boldsymbol{A}' - \boldsymbol{A}\| + \|\boldsymbol{B}' - \boldsymbol{B}\| < \varepsilon$$

*This work was supported by the Ministry of Education and Science, Project MTM 2007-67812-CO2-01.

†Department of Applied Mathematics and Statistics, The University of the Basque Country, Faculty of Pharmacy, P.O. Box 450, E-01080 Vitoria-Gasteiz, Spain, E-mail addresses: juanmiguel.gracia@ehu.es, franciscoenrique.velasco@ehu.es

has an $(\boldsymbol{A}', \boldsymbol{B}')-$invariant subspace $\mathcal{M}'$ for which the inequality

$$\theta(\mathcal{M}', \mathcal{M}) \leq K(\|\boldsymbol{A}' - \boldsymbol{A}\| + \|\boldsymbol{B}' - \boldsymbol{B}\|)$$

holds.

An open problem is to characterize the Lipschitz stable subspaces of a pair. Some partial results are:

- In [4, Theorem 15.8.1, p. 468] it can be seen that when the pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable, every $(\boldsymbol{A}, \boldsymbol{B})-$invariant subspace is Lipschitz stable.
- In the paper [8] the following result was proved: if $\dim \mathcal{M} + \dim \operatorname{Im} \boldsymbol{B} \geq n$, then $\mathcal{M}$ is Lipschitz stable.

The characterization of the Lipschitz stable invariant subspaces of one linear map $\boldsymbol{A} : \mathbb{C}^n \to \mathbb{C}^n$ was given in 1978 and 1979 (see [4, Theorem 15.5.1, p. 459] and references therein). This theorem can be reformulated in the following terms.

Theorem 1.1 *Let $\boldsymbol{A} : \mathbb{C}^n \to \mathbb{C}^n$ be a linear map and let $\mathcal{M}$ be an $\boldsymbol{A}-$invariant subspace. Denote by $\boldsymbol{A}\,|_{\mathcal{M}}$ the restriction of $\boldsymbol{A}$ to $\mathcal{M}$*

$$A\,|_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}.$$

Then, $\mathcal{M}$ is $\boldsymbol{A}-$Lipschitz stable if and only if for each eigenvalue λ of $\boldsymbol{A}\,|_{\mathcal{M}}$, the equality

$$\mathcal{R}_\lambda(\boldsymbol{A}\,|_{\mathcal{M}}) = \mathcal{R}_\lambda(\boldsymbol{A})$$

holds, where by $\mathcal{R}_\lambda(\boldsymbol{A}) := \operatorname{Ker}(\lambda \boldsymbol{I} - \boldsymbol{A})^n$ we denote the root subspace *of $\boldsymbol{A}$ associated with the eigenvalue λ.*

The organization of this paper is as follows. In Section 2 the statements of the main results are given. A matrix reformulation of the concepts of invariant and Lipschitz stable subspaces are settled in Section 3. In Section 4 a matrix reformulation of the main theorems are given. The proof of these theorems is the content of Sections 5 and 6.

2 Main results

For stating the first result, we are going to make some considerations. Given a pair of linear maps $\boldsymbol{A} : \mathbb{C}^n \to \mathbb{C}^n$ and $\boldsymbol{B} : \mathbb{C}^m \to \mathbb{C}^n$, let $\mathcal{C}(\boldsymbol{A}, \boldsymbol{B}) = \operatorname{Im} \boldsymbol{B} + \operatorname{Im}(\boldsymbol{A} \circ \boldsymbol{B}) + \cdots + \operatorname{Im}(\boldsymbol{A}^{n-1} \circ \boldsymbol{B})$ be the controllability subspace of the pair $(\boldsymbol{A}, \boldsymbol{B})$. Let $\mathcal{K}$ be a subspace of $\mathbb{C}^n$ such that $\mathbb{C}^n = \mathcal{K} \oplus \mathcal{C}(\boldsymbol{A}, \boldsymbol{B})$. We will consider the projector $\pi_{\mathcal{K}}$ on $\mathcal{K}$ along $\mathcal{C}(\boldsymbol{A}, \boldsymbol{B})$

$$\pi_{\mathcal{K}} : \mathbb{C}^n \to \mathcal{K}.$$

Thus, $\operatorname{Im} \pi_{\mathcal{K}} = \mathcal{K}$, and $\operatorname{Ker} \pi_{\mathcal{K}} = \mathcal{C}(\boldsymbol{A}, \boldsymbol{B})$. Let us consider

$$\boldsymbol{A}_1 := (\pi_{\mathcal{K}} \circ A)\,|_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}, \quad \text{and} \quad \mathcal{M}_1 := \pi_{\mathcal{K}}(\mathcal{M}). \tag{2.1}$$

The subspace $\mathcal{M}_1$ is $\boldsymbol{A}_1-$invariant (see [7], p. 402, Remark 1). Therefore, we can consider the endomorphism

$$\boldsymbol{A}_1\,|_{\mathcal{M}_1} = (\pi_{\mathcal{K}} \circ \boldsymbol{A})\,\big|_{\pi_{\mathcal{K}}(\mathcal{M})} : \pi_{\mathcal{K}}(\mathcal{M}) \to \pi_{\mathcal{K}}(\mathcal{M}), \tag{2.2}$$

and the quotient endomorphism of $\boldsymbol{A}_1$ with respect to $\mathcal{M}_1$

$$\tilde{\boldsymbol{A}}_1 : \frac{\mathcal{K}}{\tilde{\mathcal{M}}_1} \to \frac{\mathcal{K}}{\tilde{\mathcal{M}}_1}. \tag{2.3}$$

Let $\mathcal{L}$ be an $(\boldsymbol{A}, \boldsymbol{B})$–invariant subspace. By [4, Theorem 6.1.1] there exists a linear map $\boldsymbol{F} : \mathbb{C}^n \to \mathbb{C}^m$ such that $\mathcal{L}$ is $(\boldsymbol{A} + \boldsymbol{BF})-$invariant. We will denote by $\Lambda(\boldsymbol{A})$ the spectrum or set of eigenvalues of $\boldsymbol{A}$.

Theorem 2.1 *Let $\boldsymbol{A} : \mathbb{C}^n \to \mathbb{C}^n$, $\boldsymbol{B} : \mathbb{C}^m \to \mathbb{C}^n$ be linear maps. Let $\mathcal{M}$ be an $(\boldsymbol{A}, \boldsymbol{B})-$invariant subspace such that $\mathcal{M} \cap \operatorname{Im} \boldsymbol{B} = \{0\}$. Then the subspace $\mathcal{N} := \mathcal{M} \cap \mathcal{C}(\boldsymbol{A}, \boldsymbol{B})$ is $(A, B)-$invariant. Let $\boldsymbol{F} : \mathbb{C}^n \to \mathbb{C}^m$ be such that $\mathcal{N}$ is $(\boldsymbol{A} + \boldsymbol{BF})-$invariant. Let us consider the restriction*

$$(\boldsymbol{A} + \boldsymbol{BF})_{|\mathcal{N}} : \mathcal{N} \to \mathcal{N}.$$

Then the subspace $\mathcal{M}$ is $(\boldsymbol{A}, \boldsymbol{B})-$Lipschitz stable if and only if

$$\left[\Lambda\left((\boldsymbol{A} + \boldsymbol{BF})_{|\mathcal{N}}\right) \cup \Lambda\left(\boldsymbol{A}_1\,|_{\mathcal{M}_1}\right)\right] \cap \Lambda\left(\tilde{\boldsymbol{A}}_1\right) = \emptyset.$$

Remark 2.1 The statement of this theorem does not depend on the choice of $\mathcal{K}$ (see [7], p. 402, Remark 2) and $\boldsymbol{F}$ (see [8], p. 18, Remark 2.1).

Before stating the second theorem we introduce some notations.

$$\begin{aligned}
&\Delta := \{\lambda \in \Lambda(\boldsymbol{A}_1) | \{0\} \neq \mathcal{R}_\lambda(\boldsymbol{A}_1) \cap \mathcal{M}_1 \neq \mathcal{R}_\lambda(\boldsymbol{A}_1)\},\\
&\mathcal{H} := \oplus_{\lambda \in \Delta} \mathcal{R}_\lambda(\boldsymbol{A}_1),\\
&\tilde{\mathcal{M}}_1 := \oplus_{\lambda \in \Delta}(\mathcal{R}_\lambda(\boldsymbol{A}_1) \cap \mathcal{M}_1).
\end{aligned}$$

$\mathcal{R}_\lambda(\boldsymbol{A}_1)$ being the *root subspace* associated to the eigenvalue λ. With these notations we have the following result.

Theorem 2.2 *Let $\boldsymbol{A} : \mathbb{C}^n \to \mathbb{C}^n$, $\boldsymbol{B} : \mathbb{C}^m \to \mathbb{C}^n$ be linear maps. Let $\mathcal{M}$ be an $(\boldsymbol{A}, \boldsymbol{B})-$invariant subspace such that $\mathcal{M} \supseteq \mathcal{C}(\boldsymbol{A}, \boldsymbol{B})$ and*

$$\dim \mathcal{H} - \dim \tilde{\mathcal{M}}_1 \leq \operatorname{rank} \boldsymbol{B}.$$

Then the subspace $\mathcal{M}$ is $(\boldsymbol{A}, \boldsymbol{B})-$Lipschitz stable.

3 Notations and preliminary results

By $\mathrm{GL}_n(\mathbb{C})$ will denote the general linear group formed by the $n \times n$ invertible matrices over $\mathbb{C}$. We are going to reformulate in matrix terms the concepts of invariant and Lipschitz stable subspace (see [8]). A linear map $\boldsymbol{A} : \mathbb{C}^n \to \mathbb{C}^n$ (or a pair of linear maps $\boldsymbol{A} : \mathbb{C}^n \to \mathbb{C}^n$ and $\boldsymbol{B} : \mathbb{C}^m \to \mathbb{C}^n$) will be identified with a matrix $A \in \mathbb{C}^{n\times n}$ (or with a pair $(A, B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$), that is its representation in some bases. Also, each subspace $\mathcal{M}$ of $\mathbb{C}^n$ can be represented by a matrix, X, called a *basis matrix*, whose columns are linearly independent and generate $\mathcal{M}$. This fact will be denoted by $\mathcal{M} = \langle X\rangle$. Note that if Y is another basis matrix of $\mathcal{M}$, then $Y = XP$ for some invertible matrix P. Taking into account these remarks, the concept of invariant subspace will be stated in the following way.

Definition 3.1 Let $(A, B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ be a pair of matrices. Let $\mathcal{M}$ be a subspace of $\mathbb{C}^n$ of dimension p. The subspace $\mathcal{M}$ is called $(A, B)-$invariant if there exist matrices $X \in \mathbb{C}^{n\times p}$, $H \in \mathbb{C}^{p\times p}$ and $U \in \mathbb{C}^{m\times p}$ such that X is a basis matrix of $\mathcal{M}$ and $AX = XH + BU$.

This definition enables us to make a reformulation of the concept of Lipschitz stability of an $(A, B)-$invariant subspace in terms of limits of sequences of matrices. To do so, we will use the following result on the convergence of a sequence of subspaces that one deduces straightforwardly from ([2], section 1.5, p. 29–31), ([4], Theorem 13.5.1) and ([3], Theorem I-2-6).

Proposition 3.1 *Let $\mathcal{M}$ be a p-dimensional subspace of $\mathbb{C}^n$. Let $\{\mathcal{M}_q\}_{q=1}^{\infty}$ be a sequence of subspaces of $\mathbb{C}^n$ that converges to $\mathcal{M}$ in the gap metric. Then, for each $X \in \mathbb{C}^{n\times p}$, basis matrix of $\mathcal{M}$, there exists a sequence of matrices $\{X_q\}_{q=1}^{\infty}$ converging to X, two positive constants K_1, K_2, and a positive integer q_0, such that for $q \geq q_0$, X_q is a basis matrix of $\mathcal{M}_q$, and*

$$K_1\|X_q - X\| \leq \theta(\mathcal{M}_q, \mathcal{M}) \leq K_2\|X_q - X\|.$$

From Definition 3.1 and Proposition 3.1, we can reformulate the concept of Lipschitz stable subspace in terms of the convergence of sequences of matrices. The result is the following one.

Proposition 3.2 *Let $(A, B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ be a pair of matrices and let $\mathcal{M}$ be an $(A, B)-$invariant subspace such that* $\dim \mathcal{M} = p$. *Then, the following two assertions are equivalent.*
(i) *$\mathcal{M}$ is $(A, B)-$Lipschitz stable.*
(ii) *For every basis matrix $X \in \mathbb{C}^{n\times p}$ of $\mathcal{M}$, and for every sequence of matrix pairs $\{(A_q, B_q)\}_{q=1}^{\infty}$ converging to (A, B) when $q \to \infty$, there are sequences of matrices $\{X_q\}_{q=1}^{\infty}$, $\{H_q\}_{q=1}^{\infty}$ and $\{U_q\}_{q=1}^{\infty}$, a constant $K > 0$, and an integer $q_0 > 1$, such that for $q \geq q_0$,*

- *X_q is a matrix of rank p,*
- *the subspace $\langle X_q\rangle$ is $(A_q, B_q)-$invariant and $A_qX_q = X_qH_q + B_qU_q$,*
- *$\|X_q - X\| \leq K(\|A_q - A\| + \|B_q - B\|)$.*

In addition, if

$$X = \begin{pmatrix} I_p \\ 0 \end{pmatrix},$$

then for $q \geq q_0$ *we can choose* X_q *such that*

$$X_q = \begin{pmatrix} I_p \\ Y_q \end{pmatrix},$$

where $Y_q \to 0$ *when* $q \to \infty$*.*

Remark 3.1 Definition 3.1 and Proposition 3.2 have their equivalents for the case of square matrices, taking $B = 0$.

We are going to see some results that will simplify the statements of the theorems in the previous section and some proofs. The first is the following one.

Proposition 3.3 *Let* $(A, B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$*. Let* $(\overline{A}, \overline{B}) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ *be a pair feedback equivalent to* (A, B)*; that is to say,*

$$(\overline{A}, \overline{B}) = (PAP^{-1} + PBF, PBQ)$$

with $P \in \mathrm{GL}_n(\mathbb{C})$*,* $Q \in \mathrm{GL}_m(\mathbb{C})$ *and* $F \in \mathbb{C}^{m\times n}$*. Let* $\mathcal{M} \subseteq \mathbb{C}^n$ *be an* $(A, B)-$*invariant subspace. Then,* $\mathcal{M}$ *is* $(A, B)-$*Lipschitz stable if and only if* $P\mathcal{M}$ *is* $(\overline{A}, \overline{B})-$*Lipschitz stable.*

Proof. First, let us suppose that $P\mathcal{M}$ is $(\overline{A}, \overline{B})-$Lipschitz stable. Let X be a basis matrix of $\mathcal{M}$, so $Y = PX$ is a basis matrix of $P\mathcal{M}$. Let $\{(A_q, B_q)\}_{q=1}^{\infty}$ be a sequence of matrix pairs that converges to (A, B). Since

$$\{(\overline{A}_q, \overline{B}_q)\}_{q=1}^{\infty} = \{PA_qP^{-1} + PB_qF, PB_qQ\}_{q=1}^{\infty} \text{ converges to } (\overline{A}, \overline{B}),$$

by Proposition 3.2, there exists a sequence of matrices $\{Y_q\}_{q=1}^{\infty}$ converging to Y and a positive integer q_0, such that for $q \geq q_0$ the subspace $\langle Y_q\rangle$ is $(\overline{A}_q, \overline{B}_q)-$invariant and

$$\|Y_q - Y\| \leq K(\|\overline{A}_q - \overline{A}\| + \|\overline{B}_q - \overline{B}\|). \tag{3.4}$$

Let $X_q := P^{-1}Y_q$. Now, for $q \geq q_0$, the subspace $\langle X_q\rangle$ is $(A_q, B_q)-$invariant, and

$$\|X_q - X\| = \|P^{-1}Y_q - P^{-1}Y\| \leq \|P^{-1}\| \, \|Y_q - Y\|.$$

By (3.4), for $q \geq q_0$,

$$\|X_q - X\| \leq K\|P^{-1}\|(\|\overline{A}_q - \overline{A}\| + \|\overline{B}_q - \overline{B}\|). \tag{3.5}$$

Since

$$\begin{aligned} \|\overline{A}_q - \overline{A}\| &= \|PA_qP^{-1} + PB_qF - PAP^{-1} - PBF\| \\ &\leq \|P\| \, \|A_q - A\| \, \|P^{-1}\| + \|P\| \, \|B_q - B\| \, \|F\| \\ &\leq K_1(\|A_q - A\| + \|B_q - B\|), \end{aligned} \tag{3.6}$$

and

$$\|\overline{B}_q - \overline{B}\| \leq \|P\| \, \|Q\| \, \|B_q - B\|, \tag{3.7}$$

introducing inequalities (3.6) and (3.7) in (3.5), we deduce that there is a constant $K_2 > 0$ such that for $q \geq q_0$,

$$\|X_q - X\| \leq K_2(\|A_q - A\| + \|B_q - B\|).$$

Hence, by Proposition 3.2, the subspace $\mathcal{M}$ is $(A, B)-$Lipschitz stable. The converse is proved in a similar way. □

By means of similar arguments to the ones used in the proofs of Proposition 3.4 of [8] and Proposition 3.3 of [5], we infer the next assertion and proposition.

Assertion 3.1 *To study the Lipschitz stability of the $(A, B)-$invariant subspaces $\mathcal{M}$ there is no loss generality if we consider that B has full rank.*

Proof. Let Q be an invertible matrix, such that $BQ = [\bar{B}, 0]$, with $\bar{B}$ of full rank. By Proposition 3.3, $\mathcal{M}$ is $(A, B)-$Lipschitz stable if and only if $\mathcal{M}$ is $(A, [\bar{B}, 0])-$Lipschitz stable. Therefore, we must prove that $\mathcal{M}$ is $(A, [\bar{B}, 0])$–Lipschitz stable if and only if $\mathcal{M}$ is $(A, \bar{B})$–Lipschitz stable.

Firstly, let us suppose that $\mathcal{M}$ is $(A, \bar{B})-$Lipschitz stable. Let us consider a sequence of matrix pairs $\{(A_q, [\bar{B}_q, \epsilon_q])\}_{q=1}^{\infty}$ converging to $(A, [\bar{B}, 0])$ when $q \to \infty$. Let X be a basis matrix of $\mathcal{M}$. Then we have $(A_q, \bar{B}_q) \to (A, \bar{B})$ when $q \to \infty$. Thus, by Proposition 3.2, there exist sequences of matrices $\{X_q\}_{q=1}^{\infty}$, $\{H_q\}_{q=1}^{\infty}$ and $\{U_q\}_{q=1}^{\infty}$, and an integer q_0 exists such that for all $q \geq q_0$, $A_q X_q = X_q H_q + \bar{B}_q U_q$, and besides $X_q \to X$ when $q \to \infty$. From here, for all $q \geq q_0$, we have $A_q X_q = X_q H_q + \bar{B}_q U_q + \epsilon_q 0$. Hence, by Proposition 3.2, the subspace $\mathcal{M}$ is $(A, [\bar{B}, 0])-$Lipschitz stable. The converse is proved in a similar way. □

Proposition 3.4 *Let $(A, B) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times m}$ be a pair of matrices and let $\mathcal{M}$ be an $(A, B)-$invariant subspace. Then the following statements are equivalent.*

(i) *The subspace $\mathcal{M}$ is $(A, B)-$Lipschitz stable.*

(ii) *For each $\lambda \in \mathbb{C}$ the subspace $\mathcal{M}$ is $(A + \lambda I_n, B)-$Lipschitz stable.*

Proof. Taking $\lambda = 0$, we prove (ii)⇒(i). In order to prove the reverse implication, let X be a basis matrix of $\mathcal{M}$, let $\lambda \in \mathbb{C}$, and let $\{C_q\}_{q=1}^{\infty}$ be a sequence that converges to $A + \lambda I_n$. Then, since $\{C_q - \lambda I_n\}_{q=1}^{\infty}$ converges to A, by Proposition 3.2, there exist sequences of matrices $\{X_q\}_{q=1}^{\infty}$ converging to X, $\{H_q\}_{q=1}^{\infty}$ and $\{U_q\}_{q=1}^{\infty}$, and a positive integer q_0, such that for all $q \geq q_0$,

$$C_q X_q = X_q(H_q + \lambda I_p) + BU_q.$$

Therefore from Proposition 3.2 we deduce that the subspace $\mathcal{M}$ is $(A + \lambda I_n, B)$–Lipschitz stable. □

For any positive integers p and q we denote by $0_{p\times q}$ the $p \times q$ zero matrix. Given a pair of matrices $(A, B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$, we denote the set of all sequences of pairs of matrices that converge to (A, B) by $\mathcal{CS}(A, B)$.

We will say that a set $\mathcal{G} \subset \mathcal{CS}(A, B)$ is a *Lipschitz generator subset of* $\mathcal{CS}(A, B)$ if for every sequence

$$\{(A_q, B_q)\}_{q=1}^{\infty} \in \mathcal{CS}(A, B),$$

there exist sequences

$$\{(\overline{A}_q, \overline{B}_q)\}_{q=1}^{\infty} \in \mathcal{G} \text{ and } \{(P_q, Q_q, F_q)\}_{q=1}^{\infty} \text{ converging to } (I_n, I_m, 0_{m\times n}),$$

and there exist a positive integer number q_0 and a constant $K > 0$, that depends on the preceding sequences, such that for $q \geq q_0$,

$$\begin{cases} A_q = P_q\overline{A}_qP_q^{-1} + P_q\overline{B}_qF_q, \\ B_q = P_q\overline{B}_qQ_q, \\ \max\{\|P_q - I_n\|, \|Q_q - I_m\|, \|F_q\|\} \leq K(\|A_q - A\| + \|B_q - B\|). \end{cases}$$

With the preceding notation we have the next proposition.

Proposition 3.5 *Let $(A, B) \in \mathbb{C}^{n\times n}\times\mathbb{C}^{n\times m}$ be a pair of matrices, and let $\mathcal{M}$ be an $(A, B)-$invariant subspace and X a basis matrix of $\mathcal{M}$. Let $\mathcal{G}$ be a Lipschitz generator subset of $\mathcal{CS}(A, B)$. Then the statements below are equivalent.*

(i) *The subspace $\mathcal{M}$ is $(A, B)-$Lipschitz stable.*

(ii) *For all sequence $\{(\overline{A}_q, \overline{B}_q)\}_{q=1}^{\infty} \in \mathcal{G}$, there exist a sequence of matrices $\{\overline{X}_q\}_{q=1}^{\infty}$ that converges to X, a constant $K_1 > 0$ and a positive integer q_1, such that for $q \geq q_1$, the subspace $\langle\overline{X}_q\rangle$ is $(\overline{A}_q, \overline{B}_q)-$invariant, and*

$$\|\overline{X}_q - X\| \leq K_1(\|\overline{A}_q - A\| + \|\overline{B}_q - B\|).$$

For proving this proposition, we need the following lemma.

Lemma 3.1 *With the preceding notation, let us suppose that*

$$\max\{\|P_q - I_n\|, \|Q_q - I_m\|\} \leq K(\|A_q - A\| + \|B_q - B\|) < 1,$$

for $q \geq q_1$. Then there are an integer $q_2 > 0$ and a constant $K_1 > 0$ such that for $q \geq q_2$,

$$\max\{\|P_q^{-1} - I_n\|, \|Q_q^{-1} - I_m\|\} \leq K_1(\|A_q - A\| + \|B_q - B\|).$$

Proof. From [Fact 9.9.42, p. 382] of [1], for any $M \in \mathbb{C}^{n\times n}$ that satisfies $\|M\| < 1$, we deduce

$$\|(I_n + M)^{-1} - I_n\| \leq \frac{\|M\|}{1 - \|M\|}. \tag{3.8}$$

Thus,

$$\|P_q^{-1} - I_n\| \leq \frac{\|P_q - I_n\|}{1 - \|P_q - I_n\|} \leq \frac{K(\|A_q - A\| + \|B_q - B\|)}{1 - \|P_q - I_n\|}.$$

Since $P_q \to I_n$ when $q \to \infty$, then there is an integer $q_2 > 0$ such that $1 - \|P_q - I_n\| \geq 1/2$ for $q \geq q_2$. This proves the lemma. □

Proof of Proposition 3.5. The implication (i) ⇒ (ii) follows from Proposition 3.2. Let $\{(A_q, B_q)\}_{q=1}^{\infty}$ be a sequence that converges to (A, B). Then there exist sequences of pairs and triples of matrices

$$\{(\overline{A}_q, \overline{B}_q)\}_{q=1}^{\infty} \in \mathcal{G} \text{ and } \{(P_q, Q_q, F_q\}_{q=1}^{\infty} \text{ converging to } (I_n, I_m, 0_{m\times n}),$$

a constant $K > 0$ and a positive integer q_0, such that for $q \geq q_0$,

$$(A_q, B_q) = (P_q\overline{A}_qP_q^{-1} + P_q\overline{B}_qF_q, P_q\overline{B}_qQ_q),$$

that is

$$(\overline{A}_q, \overline{B}_q) = (P_q^{-1}A_qP_q - B_qF_qP_q, P_q^{-1}B_qQ_q^{-1}), \tag{3.9}$$

and

$$\max\{\|P_q - I_n\|, \|Q_q - I_m\|, \|F_q\|\} \leq K(\|A_q - A\| + \|B_q - B\|). \tag{3.10}$$

Moreover, by Lemma 3.1, for $q \geq q_0$,

$$\max\{\|P_q^{-1} - I_n\|, \|Q_q^{-1} - I_m\|, \|F_q\|\} \leq K(\|A_q - A\| + \|B_q - B\|). \tag{3.11}$$

Since $A_q \to A$, $B_q \to B$, $P_q \to I_n$, and $Q_q \to I_m$, we can suppose that there is a constant $L > 0$ such that for $q \geq q_0$,

$$\max\{\|A_q\|, \|B_q\|, \|P_q\|, \|P_q^{-1}\|, \|Q_q\|, \|Q_q^{-1}\|\} \leq L. \tag{3.12}$$

By (ii), there exist a sequence of matrices $\{\overline{X}_q\}_{q=1}^{\infty}$ that converges to X, a positive integer q_1, and a constant $K > 0$, such that for $q \geq q_1$ the subspace $\langle \overline{X}_q \rangle$ is $(\overline{A}_q, \overline{B}_q)-$invariant, and

$$\|\overline{X}_q - X\| \leq K_1(\|\overline{A}_q - A\| + \|\overline{B}_q - B\|). \tag{3.13}$$

Let $q_2 := \max(q_0, q_1)$. By Proposition 3.3, for $q \geq q_2$ the subspace $\langle P_q\overline{X}_q \rangle$ is $(A_q, B_q)-$invariant and

$$\begin{aligned} \|P_q\overline{X}_q - X\| &\leq \|P_q\overline{X}_q - P_qX\| + \|P_qX - X\| \leq \\ &\leq \|P_q\| \, \|\overline{X}_q - X\| + \|P_q - I_n\| \, \|X\|. \end{aligned}$$

From (3.10), (3.11) and (3.13), for $q \geq q_2$ we infer that

$$\begin{aligned} \|P_q\overline{X}_q - X\| &\leq LK_1(\|\overline{A}_q - A\| + \|\overline{B}_q - B\|) \\ &+ \|X\|K(\|A_q - A\| + \|B_q - B\|). \end{aligned} \tag{3.14}$$

Let us observe now that by (3.9),

$$\begin{aligned}\|\overline{A}_q - A\| &= \|P_qA_qP_q^{-1} - B_qF_qP_q - A\| \leq \|P_qA_qP_q^{-1} - A\| + \|B_qF_qP_q\| \\ &\leq \|P_qA_qP_q^{-1} - A\| + \|B_q\|\,\|F_q\|\,\|P_q\|. \end{aligned} \tag{3.15}$$

As

$$\begin{aligned}\|P_qA_qP_q^{-1} - A\| &\leq \|P_qA_qP_q^{-1} - P_qA_q\| + \|P_qA_q - P_qA\| + \|P_q - A\| \\ &\leq \|P_q\|\,\|A_q\|\,\|P_q^{-1} - I_n\| + \|P_q\|\,\|A_q - A\| + \|P_q - I_n\|\,\|A\|,\end{aligned}$$

from (3.10), (3.11) and (3.12), there are an integer $q_3 > 0$ and a constant $K_3 > 0$ such that for $q \geq q_3$,

$$\|P_qA_qP_q^{-1} - A\| \leq K_3(\|A_q - A\| + \|B_q - B\|). \tag{3.16}$$

On the other hand, by (3.10) and (3.12),

$$\|B_q\|\,\|F_q\|\,\|P_q\| \leq L^2K(\|A_q - A\| + \|B_q - B\|).$$

From this inequality, (3.15) and (3.16), we conclude that there are an integer $q_4 > 0$ and a constant $K_4 > 0$ such that for $q \geq q_4$

$$\|\overline{A}_q - A\| \leq K_4(\|A_q - A\| + \|B_q - B\|). \tag{3.17}$$

Analogously, one proves that there are an integer $q_5 > 0$ and a constant $K_5 > 0$ such that for $q \geq q_5$

$$\|\overline{B}_q - B\| \leq K_5(\|A_q - A\| + \|B_q - B\|).$$

Substituting the last inequality and (3.17) in (3.14), one proves the proposition. □

Proposition 3.6 *Let $(A, B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ be a pair with B of full rank. Then the set*

$$\left\{\{(A_q, B)\}_{q=1}^{\infty} \mid \{A_q\}_{q=1}^{\infty} \text{ converges to } A\right\}$$

is a Lipschitz generator subset of $\mathcal{CS}(A, B)$.

Proof. First, since B is full rank, there exist invertible matrices P and Q such that

$$PBQ = \begin{pmatrix} I_m \\ 0 \end{pmatrix}. \tag{3.18}$$

Let

$$(\overline{A}, \overline{B}) := (PAP^{-1}, PBQ). \tag{3.19}$$

Let us consider a sequence of matrix pairs $\{(A_q, B_q)\}_{q=1}^{\infty}$ converging to (A, B). Then, the sequence

$$\{(\overline{A}_q, \overline{B}_q)\}_{q=1}^{\infty} = \{(PA_qP^{-1}, PB_qQ)\}_{q=1}^{\infty} \tag{3.20}$$

converges to $(\overline{A}, \overline{B})$. Denote sequences of matrices $\{\varepsilon_q\}_{q=1}^{\infty}$ and $\{\delta_q\}_{q=1}^{\infty}$ converging to zero such that

$$\overline{B}_q = \begin{pmatrix} I_m + \varepsilon_q \\ \delta_q \end{pmatrix}. \tag{3.21}$$

We see that there exists an integer $q_0 > 0$ such that for $q \geq q_0$, the matrix $I_m + \varepsilon_q$ is invertible. Let us define, for $q \geq q_0$,

$$P_q := \begin{pmatrix} (I_m + \varepsilon_q)^{-1} & 0 \\ -\delta_q(I_m + \varepsilon_q)^{-1} & I_{n-m} \end{pmatrix}. \tag{3.22}$$

By (3.22) and (3.21), for $q \geq q_0$, $P_q\overline{B}_q = \overline{B}$. From (3.20) and (3.19), $P_qPB_qQ = PBQ$. In consequence, for $q \geq q_0$,

$$B_q = P^{-1}P_q^{-1}PB.$$

Thus, for proving the proposition it suffices to demonstrate that there are an integer $q_1 > 0$ and a constant $K > 0$ such that for $q \geq q_1$,

$$\|P^{-1}P_q^{-1}P - I_n\| \leq K\|B_q - B\|.$$

To prove this, by (3.22), let us observe that

$$\|P^{-1}P_q^{-1}P - I_n\| \leq \|P^{-1}\| \, \|P_q^{-1} - I_n\| \, \|P\| \leq \|P^{-1}\| \, \|P\| \left\| \begin{matrix} \varepsilon_q \\ \delta_q \end{matrix} \right\|. \tag{3.23}$$

Last, from (3.18), (3.21) and (3.19) we see that

$$\left\| \begin{matrix} \varepsilon_q \\ \delta_q \end{matrix} \right\| = \|\overline{B}_q - \overline{B}\| \leq \|P\| \, \|B_q - B\| \, \|Q\|.$$

The substitution of this inequality in (3.23) ends the proof. □

The following result simplifies the checking of whether a subset of $\mathcal{CS}(A, B)$ is Lipschitz generator. The demonstration uses arguments similar to the ones of the proof of Proposition 3.5.

Proposition 3.7 *Let $(A, B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$. Let $\mathcal{G}$ be a Lipschitz generator subset of $\mathcal{CS}(A, B)$. Let $\mathcal{G}_1$ be a subset of $\mathcal{G}$ such that for every sequence*

$$\{(A_q, B_q)\}_{q=1}^{\infty} \in \mathcal{G},$$

there exist sequences

$$\{(\overline{A}_q, \overline{B}_q)\}_{q=1}^{\infty} \in \mathcal{G}_1 \text{ and } \{(P_q, Q_q, F_q\}_{q=1}^{\infty} \text{ converging to } (I_n, I_m, 0_{m\times n}),$$

and there exist a positive integer number q_0 and a constant $K > 0$, that depends on the preceding sequences, such that for $q \geq q_0$,

$$\begin{cases} A_q = P_q \overline{A}_q P_q^{-1} + P_q \overline{B}_q F_q, \\ B_q = P_q \overline{B}_q Q_q, \\ \max\{\|P_q - I_n\|, \|Q_q - I_m\|, \|F_q\|\} \leq K(\|A_q - A\| + \|B_q - B\|). \end{cases}$$

Then the set $\mathcal{G}_1$ is a Lipschitz generator subset of $\mathcal{CS}(A, B)$.

The next lemma follows from Theorem 1.1 and Theorems 17.9.2 and 17.9.3 of [4].

Lemma 3.2 *Let A and D be square complex matrix such that $\Lambda(A) \cap \Lambda(D) = \emptyset$. Let us consider the sequence*

$$\left\{ \begin{pmatrix} A_q & B_q \\ C_q & D_q \end{pmatrix} \right\}_{q=1}^{\infty} \text{ converging to } \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$

Then,
(a) *There exist a sequence of matrices $\{T_q\}_{q=1}^{\infty}$ converging to 0, and a constant $K > 0$ such that, for $q \geq q_0$,*

$$\begin{pmatrix} I & T_q \\ 0 & I \end{pmatrix} \begin{pmatrix} A_q & B_q \\ C_q & D_q \end{pmatrix} \begin{pmatrix} I & -T_q \\ 0 & I \end{pmatrix} = \begin{pmatrix} A'_q & 0 \\ C'_q & D'_q \end{pmatrix},$$

and

$$\|T_q\| \leq K \left\| \begin{pmatrix} A_q - A & B_q \\ C_q & D_q - D \end{pmatrix} \right\|.$$

(b) *There exist a sequence of matrices $\{P_q\}_{q=1}^{\infty}$ converging to I and a constant $K_1 > 0$, such that for $q \geq q_0$,*

$$P_q \begin{pmatrix} A_q & B_q \\ C_q & D_q \end{pmatrix} P_q^{-1} = \begin{pmatrix} \overline{A}_q & 0 \\ 0 & \overline{D}_q \end{pmatrix},$$

and

$$\|P_q - I\| \leq K_1 \left\| \begin{pmatrix} A_q - A & B_q \\ C_q & D_q - D \end{pmatrix} \right\|.$$

Remark 3.2 In the above results, the existence of a positive integer q_0 is required in such a way that the results are true for $q \geq q_0$. To simplify, without loss of generality, we suppose hereafter that $q_0 = 1$.

4 Matrix formulation of the main results

We can formulate the theorems of Section 2 in a matrix form with the aid of Definition 3.1 and Proposition 3.2. Moreover, by Assertion 3.1 we suppose that B has full rank.

By reformulating Theorem 2.1, we can consider, instead of the pair (A, B), another one block-similar to it by Proposition 3.3. So, first we are going to reduce (A, B) to another pair block-similar with a "more simple" shape.

Let $(A, B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ be a pair of matrices such that rank $B = m$. Let $\mathcal{M}$ be an $(A, B)-$invariant subspace such that $\mathcal{M} \cap \mathrm{Im}\ B = \{0\}$. In an analogous way to the preliminaries of [8, Theorem 4.2] we are going to consider the next bases matrices

$X_1 = B \in \mathbb{C}^{n\times m}$,

$X_3 \in \mathbb{C}^{n\times n_3}$, basis matrix of $\mathcal{N} = \mathcal{M} \cap \mathcal{C}(A, B)$,

$X_4 \in \mathbb{C}^{n\times n_4}$, such that $\mathcal{M} = \langle [X_3, X_4]\rangle$,

$X_2 \in \mathbb{C}^{n\times n_2}$, such that $\mathcal{C}(A, B) = \langle [X_1, X_2, X_3]\rangle$,

$X_5 \in \mathbb{C}^{n\times n_5}$ in such a way that $P^{-1} = [X_1, X_2, X_3, X_4, X_5] \in \mathrm{GL}_n(\mathbb{C})$.

Therefore, the pair $(\overline{A}, \overline{B}) = (PAP^{-1}, PB)$ and the transformed subspace $\overline{\mathcal{M}} = P\mathcal{M}$, have the following shape:

$$(\overline{A}, \overline{B}) = \left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ 0 & 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & A_{55} \end{pmatrix}, \begin{pmatrix} I_m \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right), \quad \overline{\mathcal{M}} = \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_{n_3} & 0 \\ 0 & I_{n_4} \\ 0 & 0 \end{pmatrix} \right\rangle .$$

Let us note the following associations

$$(\boldsymbol{A} + \boldsymbol{BF})_{|\mathcal{N}} \leftrightarrow A_{33}, \quad \boldsymbol{A}_1\,|_{\mathcal{M}_1} \leftrightarrow A_{44}, \quad \tilde{\boldsymbol{A}}_1 \leftrightarrow A_{55}.$$

Since

$$\mathcal{C}(\mathcal{A}, \mathcal{B}) = \left\langle \begin{pmatrix} I_{m+n_2+n_3} \\ 0 \end{pmatrix} \right\rangle ,$$

then (A_{22}, A_{21}) is a controllable pair, and therefore, there exists a matrix $F_1 \in \mathbb{C}^{m\times n_2}$ such that

$$\Lambda(A_{22} + A_{21}F_1) \cap [\Lambda(A_{33}) \cup \Lambda(A_{44}) \cup \Lambda(A_{55})] = \emptyset.$$

If we consider the matrices

$$P_1 = \begin{pmatrix} I_m & -F_1 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C})$$

and $F = (F_1A_{21}, F_1A_{22} - F_1A_{21}F_1, F_1A_{13}, 0, F_1A_{25}) \in \mathbb{C}^{m\times n}$, the pair $(\overline{\overline{A}}, \overline{\overline{B}}) = (P_1\overline{A}P_1^{-1} + P_1\overline{B}F, P_1\overline{B})$ and the transformed subspace $\overline{\overline{\mathcal{M}}} = P_1\overline{\mathcal{M}}$ have the same shape of $(\overline{A}, \overline{B})$ and $\overline{\mathcal{M}}$, where

$$\Lambda(A_{22}) \cap [\Lambda(A_{33}) \cup \Lambda(A_{44}) \cup \Lambda(A_{55})] = \emptyset.$$

Since $\Lambda(A_{22}) \cap \Lambda(A_{33}) = \emptyset$, it is well known that there exist a matrix $R_1 \in \mathbb{C}^{n_3 \times n_2}$ such that $A_{32} = A_{33}R_1 - R_1A_{22}$. Therefore, if

$$P_2 = \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & R_1 & I_{n_3} & 0 \\ 0 & 0 & 0 & I_{n_4+n_5} \end{pmatrix},$$

the pair $(\tilde{A}, \tilde{B}) = (P_2\overline{\overline{A}}P_2^{-1}, P_2\overline{\overline{B}})$ and the transformed subspace $\tilde{\mathcal{M}} = P_2\overline{\overline{\mathcal{M}}}$ have the shape

$$(\tilde{A}, \tilde{B}) = \left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & A_{25} \\ \tilde{A}_{31} & 0 & A_{33} & A_{34} & \tilde{A}_{35} \\ 0 & 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & A_{55} \end{pmatrix}, \begin{pmatrix} I_m \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right), \quad \tilde{\mathcal{M}} = \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_{n_3} & 0 \\ 0 & I_{n_4} \\ 0 & 0 \end{pmatrix} \right\rangle.$$

As $\Lambda(A_{22}) \cap \Lambda(A_{55}) = \emptyset$, using a similar reasoning we make zero the block A_{25} of the matrix $\tilde{A}$.

With these considerations and Proposition 3.3, we can formulate Theorem 2.1 in the following new form.

Theorem 4.1 (Restatement of Theorem 2.1) *Let us suppose that the pair* $(A, B) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times m}$ *and the* $(A, B)-$*invariant subspace* $\mathcal{M}$ *have the shape*

$$(A, B) = \left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & 0 \\ A_{31} & 0 & A_{33} & A_{34} & A_{35} \\ 0 & 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & A_{55} \end{pmatrix}, \begin{pmatrix} I_m \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right), \mathcal{M} = \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_{n_3} & 0 \\ 0 & I_{n_4} \\ 0 & 0 \end{pmatrix} \right\rangle,$$

where

- *for* $i = 2, 3$, $A_{i1} \in \mathbb{C}^{n_i \times m}$, $A_{ij} \in \mathbb{C}^{n_i \times n_j}$ *for* $j \geq 2$,
- $\Lambda(A_{22}) \cap [\Lambda(A_{33}) \cup \Lambda(A_{44}) \cup \Lambda(A_{55})] = \emptyset$.

Then, the following statements are equivalent.
(i) $\mathcal{M}$ *is* $(A, B)-$*Lipschitz stable,*
(ii) $[\Lambda(A_{33}) \cup \Lambda(A_{44})] \cap \Lambda(A_{55}) = \emptyset$.

In the same way, for reformulating Theorem 2.2, we are going to search a pair, block similar to the pair $(A, B) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times m}$, that has a more simple shape. Let us suppose that $\mathcal{M} \subseteq \mathbb{C}^n$ is an $(A, B)-$invariant subspace such that $\mathcal{M} \supset \mathcal{C}(A, B)$. Hereafter, we are going without loss of generality to assume that B has full rank (Assertion 3.1). From the preliminaries of [8, Theorem 4.3], we can formulate Theorem 2.2 in the following way.

Theorem 4.2 (Restatement of Theorem 2.2) *Let us suppose that the pair $(A,B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ and the $(A,B)-$invariant subspace $\mathcal{M}$ have the shape*

$$(A,B) = \left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ C & D & 0 & 0 & 0 \\ 0 & 0 & E & F & 0 \\ 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & H \end{pmatrix}, \begin{pmatrix} I_m \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right),$$

$$\mathcal{M} = \left\langle \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & I_{n_3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X \end{pmatrix} \right\rangle,$$

where

- *$(D,C) \in \mathbb{C}^{n_2\times n_2} \times \mathbb{C}^{n_2\times m}$ is a controllable pair, $E \in \mathbb{C}^{n_3\times n_3}$, $F \in \mathbb{C}^{n_3\times n_4}$, $G \in \mathbb{C}^{n_4\times n_4}$, $H \in \mathbb{C}^{n_5\times n_5}$, $X \in \mathbb{C}^{n_5\times s}$, and X has full column rank,*
- *$\Lambda(D) = \{0\}$, $\Lambda(E) = \Lambda(G)$, $\Lambda(D) \cap (\Lambda(E) \cup \Lambda(H)) = \emptyset$, $\Lambda(E) \cap \Lambda(H) = \emptyset$,*
- *the subspace $\langle X \rangle$ is $H-$Lipschitz stable.*

Then, if $m \geq n_4$, the subspace $\mathcal{M}$ is $(A,B)-$Lipschitz stable.

5 Proof of Theorem 4.1

In this section, we will demonstrate Theorem 4.1. First, we prove a preliminary lemma.

Lemma 5.1 *Let $(A,B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ and let $\mathcal{M}$ be an $(A,B)-$invariant subspace. Let us suppose they have the shape of Theorem 4.1. Let us assume that there exists a $\lambda_0 \in \mathbb{C}$ such that*

$$\lambda_0 \in [\Lambda(A_{33}) \cup \Lambda(A_{44})] \cap \Lambda(A_{55}).$$

Then, there exist a invertible matrix P such that

$$(PAP^{-1}, PB) = \left(\left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & 0 & 0 \\ \hline C & 0 & E & F & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ \hline C_1 & 0 & 0 & 0 & E_1 & F_1 \\ 0 & 0 & 0 & 0 & 0 & G_1 \end{array} \right), \left(\begin{array}{c} I_m \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array} \right) \right), \tag{5.24}$$

$$P\mathcal{M} = \left\langle \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline I & 0 \\ 0 & 0 \\ \hline 0 & I \\ 0 & 0 \end{array} \right) \right\rangle,$$

where E and G are upper triangular matrices, and

$$\lambda_0 \notin \Lambda \begin{pmatrix} E_1 & F_1 \\ 0 & G_1 \end{pmatrix}.$$

Proof. For $i = 3, 4, 5$, let $Q_i \in \mathrm{GL}_{n_i}(\mathbb{C})$ such that

$$Q_i A_{ii} Q_i^{-1} = \begin{pmatrix} H_i & 0 \\ 0 & K_i \end{pmatrix},$$

where $\Lambda(H_i) = \{\lambda_0\}$, $\lambda_0 \notin \Lambda(K_i)$ and H_i is an upper triangular matrix. Now, let $P_1 = \mathrm{diag}(I_{m+n_2}, Q_3, Q_4, Q_5)$. Then $(\overline{A}, \overline{B}) = (P_1 A P_1^{-1}, P_1 B)$ has the shape

$$(\overline{A}, \overline{B}) = \left(\left(\begin{array}{cc|cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline A_{31}^1 & 0 & H_3 & 0 & L_{35} & L_{36} & L_{37} & L_{38} \\ A_{32}^2 & 0 & 0 & K_3 & L_{45} & L_{46} & L_{47} & L_{48} \\ \hline 0 & 0 & 0 & 0 & H_4 & 0 & L_{57} & L_{58} \\ 0 & 0 & 0 & 0 & 0 & K_4 & L_{67} & L_{68} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & H_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_5 \end{array} \right), \left(\begin{array}{c} I_m \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array} \right) \right).$$

Let us note that $\overline{\mathcal{M}} = P_1 \mathcal{M} = \mathcal{M}$.

We will annul some blocks of the matrix $\overline{A}$, following the same reasoning used before Theorem 4.1 for overriding the blocks A_{32} and A_{25}. Since $\Lambda(H_i) \cap \Lambda(K_j) = \emptyset$ for $3 \le i, j \le 5$, we can make successively the following steps:

1) with H_5 and K_4 we annul the block L_{67},
2) with H_4 and K_3 we annul the block L_{45},
3) with H_5 and K_3 we annul the block L_{47},
4) with H_4 and K_5 we annul the block L_{58},
5) with H_3 and K_4 we annul the block L_{36},
6) with H_3 and K_5 we annul the block L_{38}.

Interchanging the blocks rows and columns of $\overline{A}$ according to the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 5 & 7 & 4 & 6 & 8 \end{pmatrix},$$

the pair $(\tilde{A}, \tilde{B})$ and the subspace $\tilde{\mathcal{M}}$, obtained from $(\overline{A}, \overline{B})$ and $\overline{\mathcal{M}}$ by means of the preceding transformations, have the following shape:

$$(\tilde{A}, \tilde{B}) = \left(\left(\begin{array}{cc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline A_{31}^1 & 0 & H_3 & L_{35} & L_{37} & 0 & 0 & 0 \\ 0 & 0 & 0 & H_4 & L_{57} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & H_5 & 0 & 0 & 0 \\ \hline A_{31}^2 & 0 & 0 & 0 & 0 & K_3 & L_{46} & L_{48} \\ 0 & 0 & 0 & 0 & 0 & 0 & K_4 & L_{68} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_5 \end{array} \right), \left(\begin{array}{c} I_m \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array} \right) \right)$$

$$
\tilde{\mathcal{M}} = \left\langle \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\rangle .
$$

The lemma's proof finishes by regrouping the blocks in agreement with $G = H_5$, $G_1 = K_5$ and its implications. □

Proof of Theorem 4.1. (ii)⇒(i) We are going to prove that the subspace $\mathcal{M}$ is $(A, B)-$Lipschitz stable. Let $\lambda_1 \in \mathbb{C}$ be such that

$$
\lambda_1 \notin \bigcup_{i=2}^{5} \Lambda(A_{ii}),
$$

and let $F = [\lambda_1 I_m, 0] \in \mathbb{C}^{m\times n}$. Then, by Theorem 1.1, the subspace $\mathcal{M}$ is $(A + BF)-$Lipschitz stable. Hence, $\mathcal{M}$ is $(A + BF, B)-$Lipschitz stable and, by Proposition 3.3, $\mathcal{M}$ is $(A, B)-$Lipschitz stable.

(i)⇒(ii) Let us suppose that $[\Lambda(A_{33}) \cup \Lambda(A_{44})] \cap \Lambda(A_{55}) \neq \emptyset$. Let $\lambda_0 \in \mathbb{C}$ be such that

$$
\lambda_0 \in [\Lambda(A_{33}) \cup \Lambda(A_{44})] \cap \Lambda(A_{55}).
$$

Then, by Proposition 3.3, we can assume that (A, B) and $\mathcal{M}$ have the form (5.24).

Let us prove that the subspace $\mathcal{M}$ is not $(A, B)-$Lipschitz stable. Consider the sequence

$$
\{(A_q, B_q)\}_{q=1}^{\infty} = \left\{ \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & 0 & 0 \\ \hline C & 0 & E & F & 0 & 0 \\ 0 & 0 & H_q & G & 0 & 0 \\ \hline C_1 & 0 & 0 & 0 & E_1 & F_1 \\ 0 & 0 & 0 & 0 & 0 & G_1 \end{array} \right), \left(\begin{array}{c} I_m \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array} \right) \right\}_{q=1}^{\infty},
$$

converging to (A, B), where

$$
H_q = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1/q & 0 & \cdots & 0 \end{pmatrix}. \tag{5.25}
$$

If $\mathcal{M}$ were $(A, B)-$Lipschitz stable, by Proposition 3.2, there would exist sequences of matrices $\{X_{ij}^q\}_{q=1}^{\infty}$, $\{\Omega_{ij}^q\}_{q=1}^{\infty}$ and $\{\Gamma_j^q\}_{q=1}^{\infty}$, of adequate sizes, and a constant $K > 0$, such that for all q,

$$
\left(\begin{array}{cc|cc|cc}
0 & 0 & 0 & 0 & 0 & 0 \\
A_{21} & A_{22} & 0 & 0 & 0 & 0 \\
\hline
C & 0 & E & F & 0 & 0 \\
0 & 0 & H_q & G & 0 & 0 \\
\hline
C_1 & 0 & 0 & 0 & E_1 & F_1 \\
0 & 0 & 0 & 0 & 0 & G_1
\end{array}\right)
\left(\begin{array}{cc}
X_{11}^q & X_{12}^q \\
X_{21}^q & X_{22}^q \\
\hline
I & 0 \\
X_{41}^q & X_{42}^q \\
\hline
0 & I \\
X_{61}^q & X_{62}^q
\end{array}\right) =
$$

$$
= \left(\begin{array}{cc}
X_{11}^q & X_{12}^q \\
X_{21}^q & X_{22}^q \\
\hline
I & 0 \\
X_{41}^q & X_{42}^q \\
\hline
0 & I \\
X_{61}^q & X_{62}^q
\end{array}\right)
\begin{pmatrix} \Omega_{11}^q & \Omega_{12}^q \\ \Omega_{21}^q & \Omega_{22}^q \end{pmatrix}
+
\left(\begin{array}{c}
I_m \\
0 \\
\hline
0 \\
0 \\
\hline
0 \\
0
\end{array}\right)
\left(\Gamma_1^q, \Gamma_2^q\right), \tag{5.26}
$$

and

$$
\|X_{ij}^q\| \leq K/q. \tag{5.27}
$$

Let us observe that equality (5.26) implies for all q,

$$
\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}
\begin{pmatrix} X_{11}^q & X_{12}^q \\ X_{21}^q & X_{22}^q \end{pmatrix}
+
\begin{pmatrix} E & F \\ H_q & G \end{pmatrix}
\begin{pmatrix} I & 0 \\ X_{41}^q & X_{42}^q \end{pmatrix} =
$$

$$
= \begin{pmatrix} I & 0 \\ X_{41}^q & X_{42}^q \end{pmatrix}
\begin{pmatrix} \Omega_{11}^q & \Omega_{12}^q \\ \Omega_{21}^q & \Omega_{22}^q \end{pmatrix}. \tag{5.28}
$$

On the other hand, making operations in (5.26), we see that the sequence

$$
\left\{ \begin{pmatrix} \Omega_{11}^q & \Omega_{12}^q \\ \Omega_{21}^q & \Omega_{22}^q \end{pmatrix} \right\}_{q=1}^{\infty} =
$$

$$
\left\{ \begin{pmatrix} CX_{11}^q + E + FX_{41}^q & CX_{12}^q + FX_{42}^q \\ C_1X_{11}^q + F_1X_{61}^q & C_1X_{12}^q + E_1 + F_1X_{62}^q \end{pmatrix} \right\}_{q=1}^{\infty} \tag{5.29}
$$

converges to

$$
\begin{pmatrix} E & 0 \\ 0 & E_1 \end{pmatrix},
$$

where $\Lambda(E) \cap \Lambda(E_1) = \emptyset$. Hence, by Lemma 3.2, there exist a sequence of matrices $\{T_q\}_{q=1}^{\infty}$ converging to 0, and a constant $K_1 > 0$ such that for all q,

$$
\begin{pmatrix} I & 0 \\ -T_q & I \end{pmatrix}
\begin{pmatrix} \Omega_{11}^q & \Omega_{12}^q \\ \Omega_{21}^q & \Omega_{22}^q \end{pmatrix}
\begin{pmatrix} I & 0 \\ T_q & I \end{pmatrix}
=
\begin{pmatrix} \Omega_{11}^q + \Omega_{12}^q T_q & * \\ 0 & * \end{pmatrix}, \tag{5.30}
$$

and

$$
\|T_q\| \leq K_1 \left\| \begin{pmatrix} CX_{11}^q + FX_{41}^q & CX_{12}^q + FX_{42}^q \\ C_1X_{11}^q + F_1X_{61}^q & C_1X_{12}^q + F_1X_{62}^q \end{pmatrix} \right\|.
$$

From (5.27), we deduce that there exists a constant $K_2 > 0$, such that for all q,

$$
\|T_q\| \leq K_2/q. \tag{5.31}
$$

Moreover, from (5.28) for all q,

$$\left(\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X_{11}^q & X_{12}^q \\ X_{21}^q & X_{22}^q \end{pmatrix} + \begin{pmatrix} E & F \\ H_q & G \end{pmatrix}\begin{pmatrix} I & 0 \\ X_{41}^q & X_{42}^q \end{pmatrix}\right)\begin{pmatrix} I & 0 \\ T_q & I \end{pmatrix} = \\ = \begin{pmatrix} I & 0 \\ X_{41}^q & X_{42}^q \end{pmatrix}\begin{pmatrix} I & 0 \\ T_q & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -T_q & I \end{pmatrix}\begin{pmatrix} \Omega_{11}^q & \Omega_{12}^q \\ \Omega_{21}^q & \Omega_{22}^q \end{pmatrix}\begin{pmatrix} I & 0 \\ T_q & I \end{pmatrix}.$$

By (5.30), we conclude that

$$\begin{pmatrix} C(X_{11}^q + X_{12}^q T_q) & CX_{12}^q \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} E & F \\ H_q & G \end{pmatrix}\begin{pmatrix} I & 0 \\ X_{41}^q + X_{42}^q T_q & X_{42}^q \end{pmatrix} = \\ = \begin{pmatrix} I & 0 \\ X_{41}^q + X_{42}^q T_q & X_{42}^q \end{pmatrix}\begin{pmatrix} \Omega_{11}^q + \Omega_{12}^q T_q & * \\ 0 & * \end{pmatrix}.$$

From this equality, it is immediate that for all q,

$$H_q + G(X_{41}^q + X_{42}^q T_q) = (X_{41}^q + X_{42}^q T_q)(\Omega_{11}^q + \Omega_{12}^q T_q).$$

And, since by (5.29)

$$\Omega_{11}^q + \Omega_{12}^q T_q = CX_{11}^q + E + FX_{41}^q + (CX_{12}^q + FX_{42}^q)T_q,$$

then

$$H_q + G(X_{41}^q + X_{42}^q T_q) = (X_{41}^q + X_{42}^q T_q)E \\ + (X_{41}^q + X_{42}^q T_q)\,(CX_{11}^q + FX_{41}^q + (CX_{12}^q + FX_{42}^q)\,T_q).$$

Therefore, using inequalities (5.27) and (5.31), we infer that

$$H_q + G(X_{41}^q + X_{42}^q T_q) = (X_{41}^q + X_{42}^q T_q)E + O(1/q^2), \tag{5.32}$$

when $q \to \infty$, where O is the big O notation of Landau.

Let us denote x_{ij}^q the entries of the matrix $X_{41}^q + X_{42}^q T_q$, for $1 \le i \le r$, $1 \le j \le s$. Taking into account the form of the matrices G, E and H_q, given in (5.24) and (5.25), equality (5.32) turns into equality

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1/q & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} \lambda_0 & * & \cdots & * \\ 0 & \lambda_0 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_0 \end{pmatrix}(x_{ij}^q) = \\ (x_{ij}^q)\begin{pmatrix} \lambda_0 & * & \cdots & * \\ 0 & \lambda_0 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_0 \end{pmatrix} + O(1/q^2),$$

when $q \to \infty$. From here we get

$$1/q + \lambda_0 x_{r1}^q = x_{r1}^q \lambda_0 + O(1/q^2), \quad t \to \infty,$$

which implies $1/q = O(1/q^2)$, a contradiction. □

6 Proof of Theorem 4.2

To prove this theorem we need some preliminary results. The first is the following one.

Lemma 6.1 *Let*

$$(M,N) := \left(\begin{pmatrix} 0 & 0 \\ R_1 & R_2 \end{pmatrix}, \begin{pmatrix} I_h & 0 \\ 0 & L \end{pmatrix} \right)$$

be such that L *has full rank. Then the set*

$$\mathcal{G} = \left\{ \left\{ \left(\begin{pmatrix} 0 & 0 \\ R_1^q & R_2^q \end{pmatrix}, \begin{pmatrix} I_h & 0 \\ 0 & L \end{pmatrix} \right) \right\}_{q=1}^{\infty} \right\} \in \mathcal{CS}(M,N)$$

is a Lipschitz generator subset of $\mathcal{CS}(M,N)$*.*

Proof. First, since N has full rank, by Proposition 3.6 the set

$$\mathcal{G}_1 = \left\{ \{(M_q,N)\}_{q=1}^{\infty} \, | M_q \to M \right\}$$

is a Lipschitz generator subset of $\mathcal{CS}(M,N)$.

Second, let $\{(M_q,N)\}_{q=1}^{\infty} \in \mathcal{G}_1$. Then, for all q we see that

$$(M_q,N) = \left(\begin{pmatrix} \varepsilon_q & \delta_q \\ R_1^q & R_2^q \end{pmatrix}, \begin{pmatrix} I_h & 0 \\ 0 & L \end{pmatrix} \right).$$

For all q, let

$$\overline{M}_q = \begin{pmatrix} 0 & 0 \\ R_1^q & R_2^q \end{pmatrix}.$$

Then, $\{(\overline{M}_q,N)\}_{q=1}^{\infty} \in \mathcal{G}$. Moreover for all q, $(M_q,N) = (\overline{M}_q + NF_q, N)$, where $F_q = (\varepsilon_q, \delta_q)$. Let us remark that

$$\|F_q\| \leq \|M_q - M\|.$$

Then, by Proposition 3.7, the set $\mathcal{G}$ is a Lipschitz generator subset of $\mathcal{CS}(M,N)$. □

Lemma 6.2 *Let*

$$(M,N) := \left(\begin{pmatrix} 0 & 0 \\ R_1 & R_2 \end{pmatrix}, \begin{pmatrix} I_h & 0 \\ 0 & L \end{pmatrix} \right) \quad \text{and} \quad \mathcal{N} := \left\langle \begin{pmatrix} I_h & 0 \\ 0 & Y \end{pmatrix} \right\rangle,$$

where the matrices L *and* Y *have full column rank, and let us assume that there exists a matrix* H *such that*

$$R_1 = YH.$$

Let us denote by $\tilde{\mathcal{N}} := \langle Y \rangle$*. If* $\tilde{\mathcal{N}}$ *is* $(R_2,L)-$*Lipschitz stable, then* $\mathcal{N}$ *is* $(M,N)-$*Lipschitz stable.*

Proof. First, since the subspace $\tilde{\mathcal{N}}$ is $(R_2, L)-$Lipschitz stable, by Proposition 3.2 for every sequence $\{R_2^q\}_{q=1}^{\infty}$ that converges to R_2, there exist sequences of matrices $\{Y_q\}_{q=1}^{\infty}$, $\{H_q\}_{q=1}^{\infty}$, $\{U_q\}_{q=1}^{\infty}$ and a constant $K_1 > 0$, such that for all q,

$$R_2^q Y_q = Y_q H_q + L U_q, \tag{6.33}$$

and

$$\|Y_q - Y\| \leq K_1 \|R_2^q - R_2\|. \tag{6.34}$$

Let $\{(M_q, N_q)\}_{q=1}^{\infty}$ be a sequence that converges to (M, N). Then, by Lemma 6.1 and Proposition 3.5, there is no loss of generality if for all q the pair (M_q, N_q) has the shape

$$(M_q, N_q) = \left(\begin{pmatrix} 0 & 0 \\ R_1^q & R_2^q \end{pmatrix}, \begin{pmatrix} I_h & 0 \\ 0 & L \end{pmatrix} \right).$$

Let α be a complex number such that $\alpha \notin \Lambda(R_2)$. By Proposition 3.2, to prove that $\mathcal{N}$ is $(M, N)-$Lipschitz stable it suffices to find a sequence of matrices $\{Z_q\}_{q=1}^{\infty}$ and a positive constant K, such that for all q

$$\begin{pmatrix} 0 & 0 \\ R_1^q & R_2^q \end{pmatrix} \begin{pmatrix} I_h & 0 \\ Z_q & Y_q \end{pmatrix} = \begin{pmatrix} I_h & 0 \\ Z_q & Y_q \end{pmatrix} \begin{pmatrix} \alpha I_h & 0 \\ H & H_q \end{pmatrix} + \begin{pmatrix} I_h & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} -\alpha I_h & 0 \\ 0 & U_q \end{pmatrix},$$

and

$$\|(Z_q, Y_q) - (0, Y)\| \leq K \|(R_1^q, R_2^q) - (R_1, R_2)\|.$$

From (6.33) and (6.34), if we can choose the Z_q satisfying

$$R_1^q + R_2^q Z_q = \alpha Z_q + Y_q H \tag{6.35}$$

and

$$\|Z_q\| \leq K \|(R_1^q, R_2^q) - (R_1, R_2)\|, \tag{6.36}$$

then the subspace $\mathcal{N}$ is $(M, N)-$Lipschitz stable.

Taking $Z_q := (\alpha I - R_2^q)^{-1}(R_1^q - Y_q H)$, we have equality (6.35). Lastly, since

$$R_1^q - Y_q H = (R_1^q - Y H) + (Y - Y_q) H = (R_1^q - R_1) + (Y - Y_q) H,$$

from (6.34), we deduce that for all q

$$\|R_1^q - Y_q H\| \leq K_2 \|(R_1^q, R_2^q) - (R_1, R_2)\|,$$

and therefore

$$\|Z_q\| \leq \|(\alpha I - R_q)^{-1}\| \, \|R_1^q - Y_q H\| \leq K_2 \|(\alpha I - R_q)^{-1}\| \|(R_1^q, R_2^q) - (R_1, R_2)\|.$$

This proves (6.36). □

In the following theorem, using the notations of Theorem 4.2, let us assume that $m \geq n_4$. We partition the matrix C into blocks

$$C = (C_1, C_2), \quad \text{where} \quad C_1 \in \mathbb{C}^{n_2 \times (m - n_4)}, C_2 \in \mathbb{C}^{n_2 \times n_4}.$$

With these considerations, we have the following result.

Theorem 6.1 *Let us suppose that the pair* $(\overline{A},\overline{B})$ *and the* $(\overline{A},\overline{B})-$*invariant subspace* $\overline{\mathcal{M}}$ *have the shape*

$$(\overline{A},\overline{B}) = \left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ C_2 & D & 0 & 0 & 0 \\ 0 & 0 & E & F & 0 \\ 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & H \end{pmatrix}, \begin{pmatrix} I_{n_4} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right), \overline{\mathcal{M}} = \left\langle \begin{pmatrix} I_{n_4} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle$$

where

- $C_2 \in \mathbb{C}^{n_2\times n_4}$, $D \in \mathbb{C}^{n_2\times n_2}$, $E \in \mathbb{C}^{n_3\times n_3}$, $F \in \mathbb{C}^{n_3\times n_4}$, $G \in \mathbb{C}^{n_4\times n_4}$, $H \in \mathbb{C}^{n_5\times n_5}$,
- $\Lambda(D) = \{0\}$, $\Lambda(E) = \Lambda(G)$, $\Lambda(D)\cap(\Lambda(E)\cup\Lambda(H)) = \emptyset$, $\Lambda(E)\cap\Lambda(H) = \emptyset$.

Then, the subspace $\overline{\mathcal{M}}$ *is* $(\overline{A},\overline{B})-$*Lipschitz stable.*

To prove it, we need a previous lemma.

Lemma 6.3 *With the notations of the preceding theorem, the set of sequences*

$$\{\{(\overline{A}_q,\overline{B}_q)\}_{q=1}^{\infty}\} = \left\{\left\{\left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ C_2^q & D_q & 0 & 0 & 0 \\ A_{31}^q & 0 & E_q & F_q & 0 \\ A_{41}^q & 0 & A_{43}^q & G_q & 0 \\ A_{51}^q & 0 & 0 & 0 & H_q \end{pmatrix}, \begin{pmatrix} I_{n_4} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right)\right\}_{q=1}^{\infty}\right\}$$

that converge to $(\overline{A},\overline{B})$*, is a Lipschitz generator subset of* $\mathcal{CS}(\overline{A},\overline{B})$.

Proof. Using the same reasoning as in the proof of Lemma 6.1, the set of sequences

$$\{\{(\overline{A}_q,\overline{B}_q)\}_{q=1}^{\infty}\} = \left\{\left\{\left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \overline{C}_2^q & \overline{D}_q & \overline{A}_{23}^q & \overline{A}_{24}^q & \overline{A}_{25}^q \\ \overline{A}_{31}^q & \overline{A}_{32}^q & \overline{E}_q & \overline{F}_q & \overline{A}_{35}^q \\ \overline{A}_{41}^q & \overline{A}_{42}^q & \overline{A}_{43}^q & \overline{G}_q & \overline{A}_{45}^q \\ \overline{A}_{51}^q & \overline{A}_{52}^q & \overline{A}_{53}^q & \overline{A}_{54}^q & \overline{H}_q \end{pmatrix}, \begin{pmatrix} I_{n_4} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right)\right\}_{q=1}^{\infty}\right\}$$

that converge to $(\overline{A},\overline{B})$, is a Lipschitz generator subset of $\mathcal{CS}(\overline{A},\overline{B})$.

As $\Lambda(E) = \Lambda(G)$, $\Lambda(D)\cap(\Lambda(E)\cup\Lambda(H)) = \emptyset$ and $\Lambda(E)\cap\Lambda(H) = \emptyset$, by Lemma 3.2 there exist a sequence of matrices $\{P_q\}_{q=1}^{\infty}$ and a constant $K > 0$ such that for all q

$$P_q \begin{pmatrix} \overline{D}_q & \overline{A}_{23}^q & \overline{A}_{24}^q & \overline{A}_{25}^q \\ \overline{A}_{32}^q & \overline{E}_q & \overline{F}_q & \overline{A}_{35}^q \\ \overline{A}_{42}^q & \overline{A}_{43}^q & \overline{G}_q & \overline{A}_{45}^q \\ \overline{A}_{52}^q & \overline{A}_{53}^q & \overline{A}_{54}^q & \overline{H}_q \end{pmatrix} P_q^{-1} = \begin{pmatrix} D_q & 0 & 0 & 0 \\ 0 & E_q & F_q & 0 \\ 0 & A_{43}^q & G_q & 0 \\ 0 & 0 & 0 & H_q \end{pmatrix},$$

and

$$\|P_q - I\| \le K \left\| \begin{matrix} \overline{D}_q - D & \overline{A}^q_{23} & \overline{A}^q_{24} & \overline{A}^q_{25} \\ \overline{A}^q_{32} & \overline{E}_q - E & \overline{F}_q - F & \overline{A}^q_{35} \\ \overline{A}^q_{42} & \overline{A}^q_{43} & \overline{G}_q - G & \overline{A}^q_{45} \\ \overline{A}^q_{52} & \overline{A}^q_{53} & \overline{A}^q_{54} & \overline{H}_q - H \end{matrix} \right\|.$$

To finish the proof it suffices to apply Proposition 3.7. □

Proof of Theorem 6.1. Consider an arbitrary sequence $\{(\overline{A}_q, \overline{B}_q)\}_{q=1}^{\infty}$ converging to $(\overline{A}, \overline{B})$. Then, by Proposition 3.7, there is no loss of generality if $(\overline{A}_q, \overline{B}_q)$ has the shape in Lemma 6.3. Therefore, from Proposition 3.5, to prove that $\overline{\mathcal{M}}$ is $(\overline{A}, \overline{B})-$Lipschitz stable it suffices to find sequences of matrices, of suitable sizes, $\{Z_q\}_{q=1}^{\infty}$, $\{Z_i^q\}_{q=1}^{\infty}$, $\{\Omega_{ij}^q\}_{q=1}^{\infty}$, $\{\Gamma_i^q\}_{q=1}^{\infty}$, $1 \le i, j \le 3$, an integer $q_0 > 0$ and a positive constant K, such that for $q \ge q_0$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ C_2^q & D_q & 0 & 0 & 0 \\ A_{31}^q & 0 & E_q & F_q & 0 \\ A_{41}^q & 0 & A_{43}^q & G_q & 0 \\ A_{51}^q & 0 & 0 & 0 & H_q \end{pmatrix} \begin{pmatrix} I_{n_4} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \\ Z_q & 0 & 0 \\ Z_1^q & Z_2^q & Z_3^q \end{pmatrix}$$
$$= \begin{pmatrix} I_{n_4} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \\ Z_q & 0 & 0 \\ Z_1^q & Z_2^q & Z_3^q \end{pmatrix} \begin{pmatrix} \Omega_{11}^q & \Omega_{12}^q & \Omega_{13}^q \\ \Omega_{21}^q & \Omega_{22}^q & \Omega_{23}^q \\ \Omega_{31}^q & \Omega_{32}^q & \Omega_{33}^q \end{pmatrix} + \begin{pmatrix} I_{n_4} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \Gamma_1^q & \Gamma_2^q & \Gamma_3^q \end{pmatrix} \tag{6.37}$$

and

$$\left\| \begin{pmatrix} Z_q & 0 & 0 \\ Z_1^q & Z_2^q & Z_3^q \end{pmatrix} \right\| \le K \|\overline{A}_q - \overline{A}\|. \tag{6.38}$$

Now, operating in (6.37), it follows that $\Gamma_i^q = -\Omega_{1i}^q$, $1 \le i \le 3$;

$$\begin{pmatrix} \Omega_{21}^q & \Omega_{22}^q & \Omega_{23}^q \\ \Omega_{31}^q & \Omega_{32}^q & \Omega_{33}^q \end{pmatrix} = \begin{pmatrix} C_2^q & D_q & 0 \\ A_{31}^q + F_q Z_q & 0 & E_q \end{pmatrix};$$

and

$$\begin{cases} A_{41}^q + G_q Z_q = Z_q \Omega_{11}^q \\ 0 = Z_q \Omega_{12}^q \\ A_{43}^q = Z_q \Omega_{13}^q \\ A_{51}^q + H_q Z_1^q = Z_1^q \Omega_{11}^q + Z_2^q C_2^q + Z_3^q (A_{31}^q + F_q Z_q) \\ H_q Z_2^q = Z_1^q \Omega_{12}^q + Z_2^q D_q \\ H_q Z_3^q = Z_1^q \Omega_{13}^q + Z_3^q E_q \end{cases}. \tag{6.39}$$

Let $\eta_q = \|\overline{A}_q - \overline{A}\|$. Let us define, for all q

$$Z_q := L \eta_q I_{n_4}, \tag{6.40}$$

where L is a positive constant to be determined further. Then, from (6.39), we deduce that

$$\Omega_{11}^q = G_q + \frac{A_{41}^q}{L\eta_q}, \quad \Omega_{12}^q = 0, \quad \Omega_{13}^q = \frac{A_{43}^q}{L\eta_q},$$

and

$$\begin{pmatrix} A_{51}^q & 0 & 0 \end{pmatrix} + H_q \begin{pmatrix} Z_1^q & Z_2^q & Z_3^q \end{pmatrix} = \begin{pmatrix} Z_1^q & Z_2^q & Z_3^q \end{pmatrix} S_q, \tag{6.41}$$

where, for all q,

$$S_q = \begin{pmatrix} G_q + \dfrac{A_{41}^q}{L\eta_q} & 0 & \dfrac{A_{43}^q}{L\eta_q} \\ C_2^q & D_q & 0 \\ A_{31}^q + L\eta_q F_q & 0 & E_q \end{pmatrix}. \tag{6.42}$$

For any matrix $A = (a_{ij}) \in \mathbb{C}^{m\times n}$, we define the vec operator

$$\mathrm{vec}(A) := (a_{11}, a_{21}, \ldots, a_{m1}, a_{12}, a_{22}, \ldots, a_{m2}, \ldots, a_{1n}, a_{2n}, \ldots, a_{mn})^{\mathrm{T}}$$

where $^{\mathrm{T}}$ stands for transpose (see [1], Chapter 7, p. 439). Then, from (6.41), we deduce that

$$\mathrm{vec}\begin{pmatrix} A_{51}^q & 0 & 0 \end{pmatrix} + (I \otimes H_q)\mathrm{vec}\begin{pmatrix} Z_1^q & Z_2^q & Z_3^q \end{pmatrix} = (S_q^t \otimes I)\mathrm{vec}\begin{pmatrix} Z_1^q & Z_2^q & Z_3^q \end{pmatrix}, \tag{6.43}$$

where the symbol $\otimes$ denotes the Kronecker product of matrices.

On the other hand, as $\Lambda(H) \cap [\Lambda(D) \cup \Lambda(E) \cup \Lambda(G)] = \emptyset$, there exist a constant $L > 0$ and a positive integer q_0 such that, for $q \geq q_0$

$$\Lambda(H_q) \cap \Lambda \begin{pmatrix} G_q + \dfrac{A_{41}^q}{L\eta_q} & 0 & \dfrac{A_{43}^q}{L\eta_q} \\ C_2^q & D_q & 0 \\ A_{31}^q + L\eta_q F_q & 0 & E_q \end{pmatrix} = \emptyset.$$

Therefore, by Proposition 7.2.3 of [1] and notation (6.42), we see that, for $q \geq q_0$, the matrix $S_q^t \otimes I - I \otimes H_q$ is invertible and

$$\|(S_q^t \otimes I - I \otimes H_q)^{-1}\| \leq K_1, \tag{6.44}$$

for some constant $K_1 > 0$.

Lastly, from (6.43), for $q \geq q_0$, we deduce that

$$\mathrm{vec}\begin{pmatrix} Z_1^q & Z_2^q & Z_3^q \end{pmatrix} = (S_q^t \otimes I - I \otimes H_q)^{-1}\mathrm{vec}\begin{pmatrix} A_{51}^q & 0 & 0 \end{pmatrix}.$$

By inequality (6.44), for $q \geq q_0$

$$\|(Z_1^q, Z_2^q, Z_3^q)\| \leq K_2 \|A_{51}^q\|,$$

for some constant $K_2 > 0$. This inequality and (6.40) prove the theorem. □

We are in a position to prove Theorem 4.2.

Proof of Theorem 4.2. Given that $\Lambda(H) \cap [\Lambda(D) \cup \Lambda(E) \cup \Lambda(G)] = \emptyset$, and $\Lambda(D) = \{0\}$, since the subspace $\langle X \rangle$ is $H-$Lipschitz stable, then from Theorem 1.1 the subspace

$$\left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ X \end{pmatrix} \right\rangle \text{ is } A\text{–Lipschitz stable,}$$

and thus $(A, B)-$Lipschitz stable.

By Theorem 6.1, the subspace $\overline{\mathcal{M}}$ is $(\overline{A}, \overline{B})-$Lipschitz stable. From Lemma 6.2, the subspace

$$\left\langle \begin{pmatrix} I_m & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle$$

is $(A, B)-$Lipschitz stable. This proves the theorem. □

In the light of this Theorem, an immediate question arises: What happens if $m < n_4$? In this case we cannot assure anything. To illustrate this let us see two examples.

Example 1. Let $(A, B) \in \mathbb{C}^{4\times 4} \times \mathbb{C}^{4\times 1}$ be the pair

$$(A, B) = \left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right),$$

and let $\mathcal{M}$ be the $(A, B)-$invariant subspace

$$\mathcal{M} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle.$$

Notice that $m = 1 < n_4 = 2$. Let us see that the subspace $\mathcal{M}$ is not $(A, B)-$Lipschitz stable. In order to do so, consider the sequence

$$\{(A_q, B_q)\}_{q=1}^{\infty} = \left\{ \left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1/q & 0 & 0 & 0 \\ 0 & 1/q & 0 & 0 \\ 1/q & 0 & 1/q & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) \right\}_{q=1}^{\infty}$$

converging to (A, B).

If $\mathcal{M}$ were $(A, B)-$Lipschitz stable, by Proposition 3.2, there would exist sequences $\{x_q\}_{q=1}^{\infty}$, $\{y_q\}_{q=1}^{\infty}$, $\{z_q\}_{q=1}^{\infty}$, $\{t_q\}_{q=1}^{\infty}$, $\{\alpha_q\}_{q=1}^{\infty}$, $\{\beta_q\}_{q=1}^{\infty}$, $\{\sigma_q\}_{q=1}^{\infty}$, $\{\delta_q\}_{q=1}^{\infty}$, $\{\gamma_q\}_{q=1}^{\infty}$, $\{\mu_q\}_{q=1}^{\infty}$, converging to 0, and a constant $K > 0$ such that for all q

$$
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1/q & 0 & 0 & 0 \\ 0 & 1/q & 0 & 0 \\ 1/q & 0 & 1/q & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ x_q & y_q \\ z_q & t_q \end{pmatrix}
= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ x_q & y_q \\ z_q & t_q \end{pmatrix} \begin{pmatrix} \alpha_q & \beta_q \\ \sigma_q & \delta_q \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} (\gamma_q, \mu_q), \tag{6.45}
$$

and

$$
\max\{|x_q|, |y_q|, |z_q|, |t_q|\} \leq K/q. \tag{6.46}
$$

Operating in (6.45), we deduce that

$$
\gamma_q = -\alpha_q, \quad \mu_q = -\beta_q, \quad \sigma_q = \frac{1}{q}, \quad \delta_q = 0,
$$

and

$$
\begin{cases} x_q + 1 = q\alpha_q z_q + t_q \\ q\alpha_q x_q + y_q = 0 \\ y = q\beta_q z_q \\ 1 = q\beta_q x_q. \end{cases}
$$

Now, since $1 = q\beta_q x_q$, $x_q \neq 0$ and

$$
\alpha_q = \frac{-y_q}{qx_q}, \text{ and } \beta_q = \frac{1}{qx_q}.
$$

Therefore, $y_q = \dfrac{z_q}{x_q}$ and

$$
t_q = 1 + x_q + \frac{z_q^2}{x_q^2} = 1 + x_q + y_q^2. \tag{6.47}
$$

Lastly, since from (6.46) $|t_q| \leq K/q$, then by (6.47) and (6.46) for all q we deduce that

$$
K/q \geq |t_q| \geq 1 - |x_q| - |y_q|^2 \geq 1 - K/q - K^2/q^2,
$$

which is absurd. Thus, $\mathcal{M}$ cannot be $(A, B)-$Lipschitz stable.
Example 2. Let

$$
(A, B) = \left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right), \quad \mathcal{M} = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle
$$

be the corresponding pair (A, B) and $(A, B)-$invariant subspace $\mathcal{M}$. Note that condition $m \geq n_4$ fails again, because $m = 1 < n_4 = 2$. Let us see that in this example the subspace $\mathcal{M}$ is $(A, B)-$Lipschitz stable.

Reasoning as in Lemma 6.1, we see that the set of sequences

$$\{\{(\overline{A}_q, B)\}_{q=1}^{\infty}\} = \left\{\left\{\left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1+\overline{a}_{21} & \overline{a}_{22} & \overline{a}_{23} & \overline{a}_{24} & \overline{a}_{25} \\ \overline{a}_{31} & \overline{a}_{32} & \overline{a}_{33} & 1+\overline{a}_{34} & \overline{a}_{35} \\ \overline{a}_{41} & \overline{a}_{42} & \overline{a}_{43} & \overline{a}_{44} & 1+\overline{a}_{45} \\ \overline{a}_{51} & \overline{a}_{52} & \overline{a}_{53} & \overline{a}_{54} & \overline{a}_{55} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right)\right\}_{q=1}^{\infty}\right\}$$

that converge to (A, B), is a Lipschitz generator subset of $\mathcal{CS}(A, B)$. Let

$$\{P_q\}_{q=1}^{\infty} = \left\{\begin{pmatrix} 1 & \dfrac{\overline{a}_{22}}{1+\overline{a}_{21}} & \dfrac{\overline{a}_{23}}{1+\overline{a}_{21}} & \dfrac{\overline{a}_{24}}{1+\overline{a}_{21}} & \dfrac{\overline{a}_{25}}{1+\overline{a}_{21}} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\}_{q=1}^{\infty}$$

be a sequence that converges to I_4. Note that

$$\|P_q - I_4\| \leq K\|\overline{A}_q - A\|.$$

Since $P_q B = B$ and

$$P_q \overline{A}_q P_q^{-1} + P_q B F_q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} & 1+\tilde{a}_{34} & \tilde{a}_{35} \\ \tilde{a}_{41} & \tilde{a}_{42} & \tilde{a}_{43} & \tilde{a}_{44} & 1+\tilde{a}_{45} \\ \tilde{a}_{51} & \tilde{a}_{52} & \tilde{a}_{53} & \tilde{a}_{54} & \tilde{a}_{55} \end{pmatrix},$$

where $\|F_q\| \leq K\|\overline{A}_q - A\|$, by Proposition 3.7 the set of sequences

$$\left\{\left\{\left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} & 1+\tilde{a}_{34} & \tilde{a}_{35} \\ \tilde{a}_{41} & \tilde{a}_{42} & \tilde{a}_{43} & \tilde{a}_{44} & 1+\tilde{a}_{45} \\ \tilde{a}_{51} & \tilde{a}_{52} & \tilde{a}_{53} & \tilde{a}_{54} & \tilde{a}_{55} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right)\right\}_{q=1}^{\infty}\right\}$$

that converge to (A, B), is a Lipschitz generator subset of $\mathcal{CS}(A, B)$. Lastly, by using a similar reasonings to of Example 5.2 in [8] and Proposition 3.7, the set of sequences

$$\{\{(A_q, B_q)\}_{q=1}^{\infty}\} = \left\{\left\{\left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & a_q & 0 & 1 & 0 \\ 0 & b_q & 0 & 0 & 1 \\ 0 & c_q & d_q & e_q & f_q \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right)\right\}_{q=1}^{\infty}\right\}$$

that converge to (A, B), is a Lipschitz generator subset of $\mathcal{CS}(A, B)$. By Proposition 3.5, to claim that $\mathcal{M}$ is $(A, B)-$Lipschitz stable is equivalent to finding sequences $\{x_q\}$, $\{y_q\}$ and $\{z_q\}$ that converging to 0, $\{\alpha_q\}$, $\{\beta_q\}$, $\{\sigma_q\}$, $\{\delta_q\}$, and a constant $K > 0$, such that, for all q,

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & a_q & 0 & 1 & 0 \\ 0 & b_q & 0 & 0 & 1 \\ 0 & c_q & d_q & e_q & f_q \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & x_q & 0 \\ y_q & z_q & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & x_q & 0 \\ y_q & z_q & 0 \end{pmatrix} \begin{pmatrix} \alpha_q & \beta_q & \sigma_q \\ 1 & 0 & 0 \\ 0 & \delta_q & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} (-\alpha_q, -\beta_q, -\sigma_q), \tag{6.48}$$

and

$$\|(x_q, y_q, z_q)\| \leq K \left\| \begin{pmatrix} a_q & 0 & 0 & 0 \\ b_q & 0 & 0 & 0 \\ c_q & d_q & e_q & f_q \end{pmatrix} \right\|. \tag{6.49}$$

Let us choose $x_q \neq 0$ for all q. Then, making operations in (6.48), for all q we see that

$$\begin{cases} y_q = x_q, \\ z_q = -b_q, \end{cases} \tag{6.50}$$

and on the other hand,

$$\begin{cases} \alpha_q = \dfrac{f_q x_q + b_q}{x_q}, \\ \beta_q = \dfrac{e_q x_q - b_q f_q + c_q}{x_q}, \\ \sigma_q = \dfrac{d_q}{x_q}, \\ \delta_q = x_q + a_q. \end{cases}$$

Moreover, the choosing of x_q is subject to

$$0 \neq \mid x_q \mid \leq K \|A_q - A\|,$$

for some constant $K > 0$. Then, from (6.50), we conclude that

$$\|(y_q, z_q)\| = \|(x_q, b_q)\| \leq K \|A_q - A\|.$$

This proves (6.49).

7 Conclusions

Let $(A, B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$ and $\mathcal{M}$ be any (A, B)–invariant subspace. As stated in the Introduction, if the pair (A, B) is controllable, then $\mathcal{M}$ is Lipschitz stable. The same is true if $\dim \mathcal{M} + \dim \operatorname{Im} B \geq n$, regardless of whether the pair is controllable or not.

This article addresses the problem of stability in the Lipschitz sense of the subspaces (A, B)–invariant $\mathcal{M}$ in two cases. When $\mathcal{M} \cap \operatorname{Im} B = \{0\}$, we characterize completely the subspaces that are Lipschitz stable. In the case where the subspace $\mathcal{M}$ contains the controllability subspace of the pair (A, B), we give sufficient conditions, which are not necessary, for stability in the Lipschitz sense.

In general, the problem of the characterization of (A, B)–invariant subspaces that are Lipschitz stable is open.

Acknowledgment

We thank the referee for the detailed commentaries of the first version of this paper, which have permitted to improve it.

References

[1] D. S. Bernstein: *Matrix Mathematics.* Princeton University Press, 2005.

[2] F. Chatelin: *Valeurs Propres de Matrices*, Masson, Paris, 1988.

[3] J. Ferrer, M.I. García, F. Puerta: Differentiable Families of Subspaces, *Linear Algebra Appl.*, **199** (1994) 229–252.

[4] I. Gohberg, P. Lancaster, L. Rodman: *Invariant Subspaces of Matrices with Applications*, John Wiley, New York, 1986.

[5] J.M. Gracia, F.E. Velasco: Stability of controlled invariant subspaces, *Linear Algebra Appl.*, **418** (2006) 416–434.

[6] A.C.M. Ran, L. Rodman: A class of robustness problems in matrix analysis, *Operator Theory: Advances and Applications*, **134** (2002) 337–383.

[7] L. Rodman: Stable invariant subspaces modulo a subspace, *Operator Theory: Advances and Applications*, **19** (1986) 399–413.

[8] F.E. Velasco: Stable subspaces of matrix pairs, *Linear Algebra Appl.*, **301** (1999) 15–49.