

Nearest southeast submatrix that makes multiple an eigenvalue of the normal northwest submatrix *

Juan-Miguel Gracia, Francisco E. Velasco[†]

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Abstract

Let A, B, C, D be four complex matrices, where $D \in \mathbb{C}^{m \times m}$ and $A \in \mathbb{C}^{n \times n}$ is a normal matrix. Let z_0 be an fixed eigenvalue of A . We find the distance (with respect to the spectral norm) from D to the set of matrices $X \in \mathbb{C}^{m \times m}$ such that z_0 is a multiple eigenvalue of the matrix

$$\begin{pmatrix} A & B \\ C & X \end{pmatrix}.$$

We also give an expression for one of the closest matrices.

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1 Introduction

In [4] the authors found the minimum value of $\|X - D\|$ when $X \in \mathbb{C}^{m \times m}$ is subject to the condition that z_0 is a multiple eigenvalue of the matrix

$$\begin{pmatrix} A & B \\ C & X \end{pmatrix}.$$

where $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times m}$, $C \in \mathbb{C}^{m \times n}$ and $D \in \mathbb{C}^{m \times m}$, and z_0 is a complex number which is not an eigenvalue of the matrix A . In this paper we deal with the same problem when A is a normal matrix and z_0 is an eigenvalue of A . When z_0 is not an eigenvalue of A the solution of the Problem involves matrices of polynomials in a real variable t and the inverse of square nonsingular matrices; the case when z_0 is an eigenvalue of A requires matrices of rational functions in t with a pole at $t = 0$ and the Moore–Penrose inverse instead.

We denote by $\|\cdot\|$ the matrix spectral norm. The spectrum of a square complex matrix M is denoted by $\Lambda(M)$. If $\lambda_0 \in \Lambda(M)$ the algebraic multiplicity of λ_0 is denoted by $m(\lambda_0, M)$. For a matrix $N \in \mathbb{C}^{p \times q}$ we denote by $\sigma_1(N) \geq$

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[†]Department of Applied Mathematics and Statistics, The University of the Basque Country, Faculty of Pharmacy, P.O. Box 450, E-01080 Vitoria-Gasteiz, Spain, E-mail addresses: juanmiguel.gracia@ehu.es, franciscoenrique.velasco@ehu.es

$\sigma_2(N) \geq \dots$ its singular values, and by N^\dagger its Moore–Penrose inverse. For a matrix X , we denote by $\text{Im}(X)$ and $\text{Ker}(X)$ its image and kernel subspaces. By O we denote the zero matrix of adequate size.

Moreover, as in [4], we can assume without loss of generality that $z_0 = 0$. Thus the problem we are going to solve, can be set as follows: Find the minimum

$$\min_{\substack{X \in \mathbb{C}^{m \times m} \\ \text{m}(0, \begin{pmatrix} A & B \\ C & X \end{pmatrix}) \geq 2}} \|X - D\|. \quad (1)$$

where $A \in \mathbb{C}^{n \times n}$ is a singular normal matrix $B \in \mathbb{C}^{n \times m}$, $C \in \mathbb{C}^{m \times n}$ and $D \in \mathbb{C}^{m \times m}$.

If for all $X \in \mathbb{C}^{m \times m}$ it happens that $\text{m}(0, \begin{pmatrix} A & B \\ C & X \end{pmatrix}) \leq 1$, we agree to say that the minimum distance (1) is infinite. Note that this case is possible without more that to consider $A = O \in \mathbb{C}^{n \times n}$ and $B = C = I_n$ for example, since for each $X \in \mathbb{C}^{n \times n}$ the matrix

$$\begin{pmatrix} O & I_n \\ I_n & X \end{pmatrix}$$

is nonsingular. In Section 4, the cases in which this distance is infinite will be determined.

To simplify we denote by $L_{n,m}$ the Cartesian product $\mathbb{C}^{n \times n} \times \mathbb{C}^{n \times m} \times \mathbb{C}^{m \times n}$. For a triple of matrices $\alpha := (A, B, C) \in L_{n,m}$, and for $X \in \mathbb{C}^{m \times m}$ we denote

$$M(\alpha, X) := \begin{pmatrix} A & B \\ C & X \end{pmatrix}.$$

A lower bound of the minimum (1) was given in [4]. We will remember the notations that appear in [4, (11) and (12)] to recall this bound, and for their use in this paper: Given a triple $\alpha := (A, B, C) \in L_{n,m}$ and a matrix $D \in \mathbb{C}^{m \times m}$, we define for $t \in \mathbb{R}$,

$$\begin{aligned} \rho_\alpha(t) &:= \text{rank} \left(\begin{array}{cc|cc} A & tI_n & B & O \\ O & A & O & B \end{array} \right) + \text{rank} \left(\begin{array}{cc} A & tI_n \\ O & A \\ C & O \\ O & C \end{array} \right) - \text{rank} \begin{pmatrix} A & tI_n \\ O & A \end{pmatrix}, \\ p_\alpha(t) &:= 2n + 2m - 2 - \rho_\alpha(t), \end{aligned} \quad (2)$$

$$M_\alpha(t) := \left(I_{2n} - \begin{pmatrix} A & tI_n \\ O & A \end{pmatrix} \begin{pmatrix} A & tI_n \\ O & A \end{pmatrix}^\dagger \right) \begin{pmatrix} B & O \\ O & B \end{pmatrix} \quad (3)$$

$$N_\alpha(t) := \begin{pmatrix} C & O \\ O & C \end{pmatrix} \left(I_{2n} - \begin{pmatrix} A & tI_n \\ O & A \end{pmatrix}^\dagger \begin{pmatrix} A & tI_n \\ O & A \end{pmatrix} \right) \quad (4)$$

$$\begin{aligned} S_2^\alpha(t, D) &:= (I_{2m} - N_\alpha(t)N_\alpha(t)^\dagger) \times \\ &\quad \left(\begin{pmatrix} D & tI_m \\ O & D \end{pmatrix} - \begin{pmatrix} C & O \\ O & C \end{pmatrix} \begin{pmatrix} A & tI_n \\ O & A \end{pmatrix}^\dagger \begin{pmatrix} B & O \\ O & B \end{pmatrix} \right) \\ &\quad \times (I_{2m} - M_\alpha(t)^\dagger M_\alpha(t)). \end{aligned} \quad (5)$$

We agree to write $\sup_{t \geq 0} f(t) = \infty$ if the function $f: [0, \infty) \rightarrow \mathbb{R}$ is not bounded above. This notation can also happen when $f: [0, \infty) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is regarded as a function valued in the extended real line. Then the announced lower bound of (1) is given below.

Proposition 1 ([4], Proposition 23).

$$\sup_{t \geq 0} \sigma_{p_\alpha(t)+1}(S_2^\alpha(t, D)) \leq \min_{\substack{X \in \mathbb{C}^{m \times m} \\ m(0, M(\alpha, X)) \geq 2}} \|X - D\|. \quad (6)$$

where

$$\sigma_j(S_2^\alpha(t, D)) := \begin{cases} \infty & \text{if } j < 1, \\ 0 & \text{if } j > 2m. \end{cases}$$

The aim of this paper is to prove that when A is normal and singular, the inequality (6) becomes an equality. Specifically, we prove the following result.

Theorem 2. Let $\alpha := (A, B, C) \in L_{n,m}$ be a triple of matrices, where A is normal and singular. Let $D \in \mathbb{C}^{m \times m}$. With the preceding notations, we have

$$\sup_{t > 0} \sigma_{p_\alpha(t)+1}(S_2^\alpha(t, D)) = \min_{\substack{X \in \mathbb{C}^{m \times m} \\ m(0, M(\alpha, X)) \geq 2}} \|X - D\|. \quad (7)$$

Remark 1. Let us note that in this theorem we put $t > 0$ instead of $t \geq 0$, without loss of generality.

The organization of this work is the following one. In Section 2 we give a simplified expression for $S_2^\alpha(t, D)$ and we reformulate Theorem 2 in Theorem 5. In Section 3 we introduce the auxiliary results we are going to use in this work. We analyze the asymptotic behavior of the singular values of $S_2^\alpha(t, D)$, both for $t \rightarrow 0^+$ and $t \rightarrow \infty$, and we establish the existence of the limits

$$\lim_{t \rightarrow 0^+} \sigma_{p_\alpha(t)+1}(S_2^\alpha(t, D)) \quad \text{and} \quad \lim_{t \rightarrow \infty} \sigma_{p_\alpha(t)+1}(S_2^\alpha(t, D)),$$

in Section 4. From that point on, we prove Theorem 5 throughout the following sections until the end of Section 8. Namely, in Section 5 we calculate the minimum (1) when the supremum

$$\sup_{t > 0} \sigma_{p_\alpha(t)+1}(S_2^\alpha(t, D))$$

is reached at a point t_0 such that $0 < t_0 < \infty$ and we prove equality (7). In Section 6 we study the case when

$$\sup_{t > 0} \sigma_{p_\alpha(t)+1}(S_2^\alpha(t, D)) = \lim_{t \rightarrow \infty} \sigma_{p_\alpha(t)+1}(S_2^\alpha(t, D)),$$

and, in Sections 7 and 8 we consider the case when

$$\sup_{t > 0} \sigma_{p_\alpha(t)+1}(S_2^\alpha(t, D)) = \lim_{t \rightarrow 0^+} \sigma_{p_\alpha(t)+1}(S_2^\alpha(t, D)).$$

Finally, once finished the complete proof of Theorem 5, in Section 9 we give more results that precise the equality of the supremum to the limits at 0 and ∞ , ending with a counterexample.

2 Reformulation of the main result

We denote by M^* the conjugate transpose of each complex matrix M . In this section we are going to reformulate Theorem 2, simplifying the expression of $S_2^\alpha(t, D)$ for $t > 0$ when the triple α undergoes a transformation of unitary similarity given by the unitary matrix U that diagonalizes A . For this purpose we need some properties of the Moore-Penrose inverse, which can be seen in [1, Proposition 6.1.6, p. 225].

Lemma 3. *Given a matrix $A \in \mathbb{C}^{p \times q}$, then we have*

- (1) $I_p - AA^\dagger$ and $I_q - A^\dagger A$ are orthogonal projectors.
- (2) If $S_1 \in \mathbb{C}^{p \times p}$ and $S_2 \in \mathbb{C}^{q \times q}$ are unitary, then $(S_1 A S_2)^\dagger = S_2^* A^\dagger S_1^*$.

Another result we will need is the following one.

Lemma 4. *Let $U \in \mathbb{C}^{n \times n}$ be a unitary matrix and $D \in \mathbb{C}^{m \times m}$. Then for the triple of matrices $\beta := (U^* A U, U^* B, C U) \in L_{n, m}$ and each $t > 0$ we have $S_2^\alpha(t, D) = S_2^\beta(t, D)$.*

Proof. To simplify this demonstration, we introduce the following notations:

$$L(t) = \begin{pmatrix} D & tI_m \\ O & D \end{pmatrix}, V = \begin{pmatrix} U & O \\ O & U \end{pmatrix}, F(t) = \begin{pmatrix} A & tI_n \\ O & A \end{pmatrix}, F_1(t) = V^* F(t) V,$$

$$G = \begin{pmatrix} B & O \\ O & B \end{pmatrix}, G_1 = V^* G, H = \begin{pmatrix} C & O \\ O & C \end{pmatrix}, H_1 = H V.$$

First, as the matrix V is unitary, by Lemma 3, we deduce that $(V^* F(t) V)^\dagger = V^* F(t)^\dagger V$. Hence, from (3) and (4), we obtain

$$M_\beta(t) = (I_{2n} - F_1(t) F_1(t)^\dagger) G_1 = (I_{2n} - V^* F(t) V V^* F(t)^\dagger V) V^* G = V^* M_\alpha(t),$$

$$N_\beta(t) = H_1 (I_{2n} - F_1(t)^\dagger F_1(t)) = H V (I_{2n} - V^* F(t)^\dagger V V^* F(t) V) = N_\alpha(t) V.$$

Similarly, as the matrix V is unitary, we see that $(V^* M_\alpha(t))^\dagger = M_\alpha(t)^\dagger V$ and $(N_\alpha(t) V)^\dagger = V^* N_\alpha(t)^\dagger$. Therefore,

$$I_{2m} - N_\beta(t) N_\beta(t)^\dagger = I_{2m} - N_\alpha(t) N_\alpha(t)^\dagger,$$

$$I_{2m} - M_\beta(t)^\dagger M_\beta(t) = I_{2m} - M_\alpha(t)^\dagger M_\alpha(t).$$

Finally, from $H_1 F_1(t) G_1 = H F(t) G$, by (5), we infer that

$$S_2^\beta(t, D)$$

$$= (I_{2m} - N_\alpha(t) N_\alpha(t)^\dagger) (L(t) - H F(t) G) (I_{2m} - M_\alpha(t)^\dagger M_\alpha(t))$$

$$= S_2^\alpha(t, D).$$

□

Remark 2. Let us note that if $\alpha = (A, B, C)$ and $\beta = (U^* A U, U^* B, C U)$ are two triples of matrices of $L_{n, m}$ with U unitary, then 0 is a multiple eigenvalue of $M(\alpha, X)$ if and only if it is a multiple eigenvalue of $M(\beta, X)$. Hence, by the previous lemma, in the proof of Theorem 2 there is no loss of generality if we consider the triple of matrices β .

Now, we are going to apply Lemma 4 to compute $S_2^\alpha(t, D)$. As the matrix A is normal, let $U \in \mathbb{C}^{n \times n}$ be a unitary matrix such that

$$U^*AU = \begin{pmatrix} O & O \\ O & \Sigma \end{pmatrix},$$

where $\Sigma \in \mathbb{C}^{n_2 \times n_2}$, $1 \leq n_2 < n$, is an invertible diagonal matrix. So, it is understood that $A \neq O$; the case when $A = O$ will be considered later in Remark 4. Let us consider the partition $n = n_1 + n_2$ in block matrices:

$$\left(\begin{array}{cc|c} U^*AU & U^*B & \\ \hline CU & D & \end{array} \right) = \left(\begin{array}{cc|c} O & O & B_1 \\ O & \Sigma & B_2 \\ \hline C_1 & C_2 & D \end{array} \right), \quad B_1 \in \mathbb{C}^{n_1 \times m}, C_1 \in \mathbb{C}^{m \times n_1}. \quad (8)$$

By Lemma 4, $S_2^\alpha(t, D) = S_2^\beta(t, D)$, where $\beta := (U^*AU, U^*B, CU)$. We will compute $S_2^\beta(t, D)$ for $t > 0$.

First, let us call

$$F(t) := \begin{pmatrix} O & O & tI_{n_1} & O \\ O & \Sigma & O & tI_{n_2} \\ O & O & O & O \\ O & O & O & \Sigma \end{pmatrix};$$

therefore,

$$F(t)^\dagger = \begin{pmatrix} O & O & O & O \\ O & \Sigma^{-1} & O & -t\Sigma^{-2} \\ t^{-1}I_{n_1} & O & O & O \\ O & O & O & \Sigma^{-1} \end{pmatrix},$$

and

$$F(t)F(t)^\dagger = \begin{pmatrix} I_{n_1} & O & O & O \\ O & I_{n_2} & O & O \\ O & O & O & O \\ O & O & O & I_{n_2} \end{pmatrix}, \quad F(t)^\dagger F(t) = \begin{pmatrix} O & O & O & O \\ O & I_{n_2} & O & O \\ O & O & I_{n_1} & O \\ O & O & O & I_{n_2} \end{pmatrix}.$$

Hence, from (3) and (4),

$$M_\beta(t) = \begin{pmatrix} O & O \\ O & O \\ O & B_1 \\ O & O \end{pmatrix}, \quad N_\beta(t) = \begin{pmatrix} C_1 & O & O & O \\ O & O & O & O \end{pmatrix}.$$

Consequently,

$$I_{2m} - N_\beta(t)N_\beta(t)^\dagger = \begin{pmatrix} I_m - C_1C_1^\dagger & O \\ O & I_m \end{pmatrix}, \quad I_{2m} - M_\beta(t)^\dagger M_\beta(t) = \begin{pmatrix} I_m & O \\ O & I_m - B_1^\dagger B_1 \end{pmatrix}.$$

Last,

$$\begin{pmatrix} C_1 & C_2 & O & O \\ O & O & C_1 & C_2 \end{pmatrix} F(t)^\dagger \begin{pmatrix} B_1 & O \\ B_2 & O \\ O & B_1 \\ O & B_2 \end{pmatrix} = \begin{pmatrix} C_2\Sigma^{-1}B_2 & -tC_2\Sigma^{-2}B_2 \\ t^{-1}C_1B_1 & C_2\Sigma^{-1}B_2 \end{pmatrix},$$

From the three last equalities and (5) we deduce that for $t > 0$, $S_2^\beta(t, D) =$

$$\begin{pmatrix} (I_m - C_1 C_1^\dagger)(D - C_2 \Sigma^{-1} B_2) & t(I_m - C_1 C_1^\dagger)(I_m + C_2 \Sigma^{-2} B_2)(I_m - B_1^\dagger B_1) \\ -t^{-1} C_1 B_1 & (D - C_2 \Sigma^{-1} B_2)(I_m - B_1^\dagger B_1) \end{pmatrix}.$$

Thus, by Lemma 4, it follows that for $t > 0$

$$S_2^\alpha(t, D) = \begin{pmatrix} P_C L_1 & t P_C L_2 P_B \\ -t^{-1} C_1 B_1 & L_1 P_B \end{pmatrix}, \quad (9)$$

where $P_C := I_m - C_1 C_1^\dagger$, $P_B := I_m - B_1^\dagger B_1$, $L_1 := D - C_2 \Sigma^{-1} B_2$ and $L_2 := I_m + C_2 \Sigma^{-2} B_2$.

We are going to describe a expression for $S_2^\alpha(t, D)$ in terms of A, B, C and D , avoiding the use of the unitary matrix U . Denoting

$$R := \begin{pmatrix} O & O \\ O & \Sigma \end{pmatrix} = U^* A U,$$

from Lemma 3, we deduce that

$$R^\dagger = U^* A^\dagger U = \begin{pmatrix} O & O \\ O & \Sigma^{-1} \end{pmatrix}. \quad (10)$$

As by (8) we have $(C_1, C_2) = C U$, from (10) we see that

$$(C_1, O) = (C_1, C_2)(I - R R^\dagger) = C U (I - U^* A U U^* A^\dagger U) = C(I - A A^\dagger) U. \quad (11)$$

Therefore

$$C_1 C_1^\dagger = (C_1, O)(C_1, O)^\dagger = C(I - A A^\dagger) (C(I - A A^\dagger))^\dagger;$$

that is,

$$P_C = I_m - C_1 C_1^\dagger = I_m - C(I - A A^\dagger) (C(I - A A^\dagger))^\dagger. \quad (12)$$

On the other hand, from (10) we have

$$(O, C_2) = (C_1, C_2) R^\dagger R = C U U^* A^\dagger U U^* A U = C A^\dagger A U. \quad (13)$$

Moreover, by (8) we have $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = U^* B$. We infer that

$$\begin{pmatrix} O \\ B_2 \end{pmatrix} = R R^\dagger \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = U^* A U U^* A^\dagger U U^* B = U^* A A^\dagger B.$$

Thus, using this last equality, (10) and (13),

$$C_2 \Sigma^{-1} B_2 = (O, C_2) \begin{pmatrix} O & O \\ O & \Sigma^{-1} \end{pmatrix} \begin{pmatrix} O \\ B_2 \end{pmatrix} = C A^\dagger A U U^* A^\dagger U U^* A A^\dagger B = C A^\dagger B.$$

Consequently, by the definition of L_1 , following (9), we find that

$$L_1 = D - C_2 \Sigma^{-1} B_2 = D - C A^\dagger B. \quad (14)$$

In a similar way, we can prove

$$L_2 = I_m + C_2 \Sigma^{-2} B_2 = I_m + C A^\dagger A^\dagger B. \quad (15)$$

Moreover, as $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = U^* B$, from (10) we see that

$$\begin{pmatrix} B_1 \\ O \end{pmatrix} = (I - R R^\dagger) \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = (I - U^* A U U^* A^\dagger U) U^* B = U^* (I - A A^\dagger) B. \quad (16)$$

Hence,

$$P_B = I_m - B_1^\dagger B_1 = I_m - ((I - A A^\dagger) B)^\dagger (I - A A^\dagger) B. \quad (17)$$

Finally, from (11) and (16), we know that

$$C_1 B_1 = (C_1, O) \begin{pmatrix} B_1 \\ O \end{pmatrix} = C (I - A A^\dagger) U U^* (I - A A^\dagger) B = C (I - A A^\dagger) B,$$

because $I - A A^\dagger$ is a projector (Lema 3 (1)). Now substituting this last equality and equalities (12), (14), (15) and (17), in (9), we conclude that for $t > 0$,

$$S_2^\alpha(t, D) = \begin{pmatrix} P_1 (D - C A^\dagger B) & t P_1 (I_m + C (A^\dagger)^2 B) P_2 \\ -t^{-1} C P_A B & (D - C A^\dagger B) P_2 \end{pmatrix},$$

where $P_A := I_n - A A^\dagger$, $P_1 := I_m - C P_A (C P_A)^\dagger$ and $P_2 := I_m - (P_A B)^\dagger P_A B$.

However, from this point on, in order to simplify the demonstration, we only consider the expression of $S_2^\alpha(t, D)$ given in (9). Moreover, by Remark 2, we can assume the triple $\alpha = (A, B, C)$ is in the form $(U^* A U, U^* B, C U)$ that was given in (8). From the definition of $p_\alpha(t)$, see (2), we infer that

$$p_\alpha(t) = 2m + n_1 - 2 - \text{rank}(B_1) - \text{rank}(C_1)$$

for $0 < t < \infty$.

From now on, we will abbreviate $S_2^\alpha(t, D)$ by $S_2(t)$. With these considerations, when $A \neq O$, Theorem 2 can be reformulated in the following way.

Theorem 5. *Let $\alpha = (A, B, C) \in L_{n,m}$ be a triple of matrices*

$$A := \begin{pmatrix} O & O \\ O & \Sigma \end{pmatrix}, \quad B := \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C := (C_1, C_2),$$

with $B_1 \in \mathbb{C}^{n_1 \times m}$, $C_1 \in \mathbb{C}^{m \times n_1}$ and $\Sigma \in \mathbb{C}^{n_2 \times n_2}$ an invertible diagonal matrix, $n_1 \geq 1$. Let us define

$$h := 2m + n_1 - 1 - \text{rank}(B_1) - \text{rank}(C_1). \quad (18)$$

Given $D \in \mathbb{C}^{m \times m}$, for $t > 0$, let us define

$$S_2(t) := \begin{pmatrix} P_C L_1 & t P_C L_2 P_B \\ -t^{-1} C_1 B_1 & L_1 P_B \end{pmatrix}, \quad (19)$$

where

$$P_C := I_m - C_1 C_1^\dagger, \quad P_B := I_m - B_1^\dagger B_1, \quad (20)$$

and

$$L_1 := D - C_2 \Sigma^{-1} B_2, \quad L_2 := I_m + C_2 \Sigma^{-2} B_2. \quad (21)$$

Then

$$\sup_{t>0} \sigma_h(S_2(t)) = \min_{\substack{X \in \mathbb{C}^{m \times m} \\ m(0, M(\alpha, X)) \geq 2}} \|X - D\|.$$

Remark 3. Let us note that if there exists a $t_1 > 0$ such that $\sigma_h(S_2(t_1)) = 0$, then $\text{rank}(S_2(t_1)) \leq h - 1$. By Section 4 of [4] we see that

$$\begin{aligned} \text{rank} \begin{pmatrix} A & t_1 I_n & B & O \\ O & A & O & B \\ C & O & D & t_1 I_m \\ O & C & O & D \end{pmatrix} &= \text{rank} \left(\begin{array}{cc|cc} A & B & t_1 I_n & O \\ C & D & O & t_1 I_m \\ \hline O & O & A & B \\ O & O & C & D \end{array} \right) \\ &= \rho_\alpha(t_1) + \text{rank}(S_2(t_1)) \leq \rho_\alpha(t_1) + h - 1. \end{aligned}$$

But, as

$$h - 1 = p_\alpha(t_1) = 2m + 2n - 2 - \rho_\alpha(t_1),$$

we infer that

$$\text{rank} \left(\begin{array}{cc|cc} A & B & t_1 I_n & O \\ C & D & O & t_1 I_m \\ \hline O & O & A & B \\ O & O & C & D \end{array} \right) \leq 2m + 2n - 2.$$

As it can be seen in [6, pages 444–445], this inequality implies that 0 is a multiple eigenvalue of $M(\alpha, D)$. Thus, by Proposition 1, Theorem 5 is already proved in this case. Therefore, *from now on we will assume that $\sigma_h(S_2(t)) > 0$ for $t > 0$.*

Remark 4. When the normal matrix A is the $n \times n$ zero matrix, the *statement* of Theorem 5 is reduced to

$$\sup_{t>0} \sigma_k \begin{pmatrix} (I_m - CC^\dagger)D & t(I_m - CC^\dagger)(I_m - B^\dagger B) \\ -t^{-1}CB & D(I_m - B^\dagger B) \end{pmatrix} = \min_{\substack{X \in \mathbb{C}^{m \times m} \\ m(0, M(\alpha, X)) \geq 2}} \|X - D\|, \quad (22)$$

where $k := 2m + n - 1 - \text{rank}(B) - \text{rank}(C)$. The proof of (22) might be done following similar reasonings to the $A \neq O$ case, replacing B_1 by B , C_1 by C , L_1 by D , L_2 by I_m , and removing Σ, B_2 and C_2 .

3 Auxiliary results

In this section we are going to introduce some results that will be used in this work. In the first one, we give some properties of the Moore-Penrose inverse, which can be seen in [1, Proposition 6.1.6, page 225; Fact 6.4.8, page 235] and [2].

Lemma 6. *Let $A \in \mathbb{C}^{p \times q}$ be a matrix. Then*

- (1) $\text{Ker}(I_p - AA^\dagger) = \text{Im}(A)$, $\text{Im}(I_p - AA^\dagger) = \text{Ker}(A^*) = \text{Ker}(A^\dagger)$.
- (2) $\text{Ker}(I_q - A^\dagger A) = \text{Im}(A^*) = \text{Im}(A^\dagger)$, $\text{Im}(I_q - A^\dagger A) = \text{Ker}(A)$.
- (3) $x \in \text{Im}(A)$ if and only if $x = AA^\dagger x$; $x \in \text{Im}(A^*)$ if and only if $x^* = x^* A^\dagger A$.
- (4) If $\text{rank}(A) = p$, then $AA^\dagger = I_p$; if $\text{rank}(A) = q$, then $A^\dagger A = I_q$.
- (5) Let $F \in \mathbb{C}^{q \times r}$. If $\text{rank}(A) = \text{rank}(F) = q$, then $(AF)^\dagger = F^\dagger A^\dagger$.

With this lemma and Lemma 3, we get the following properties for the matrices $P_C = (I_m - C_1 C_1^\dagger)$ and $P_B = (I_m - B_1^\dagger B_1)$, defined in (20).

- Lemma 7.** (1) P_C and P_B are orthogonal projectors.
(2) $\text{Ker}(P_C) = \text{Im}(C_1)$, $\text{Im}(P_C) = \text{Ker}(C_1^*) = \text{Ker}(C_1^\dagger)$.
(3) $\text{Ker}(P_B) = \text{Im}(B_1^*) = \text{Im}(B_1^\dagger)$, $\text{Im}(P_B) = \text{Ker}(B_1)$.
(4) If $\text{rank}(C_1) = \text{rank}(B_1) = n_1$, then $(C_1 B_1)^\dagger = B_1^\dagger C_1^\dagger$.

We will need in Sections 6 and 8 the following lemma.

Lemma 8 ([4], Lemma 33). Let $\{t_k\}_{k=1}^\infty$ be a sequence of real numbers which tends to ∞ when $k \rightarrow \infty$. Let $G \in \mathbb{C}^{p \times p}$ be a matrix and let $x_k, y_k \in \mathbb{C}^{p \times 1}$, $k = 1, 2, \dots$ be vector sequences such that

- (i) $\lim_{k \rightarrow \infty} G y_k = 0$,
(ii) $\sup_{k=1,2,\dots} \|t_k(x_k)^* G\| \leq T < \infty$, where T is a positive constant.

Then

$$\lim_{k \rightarrow \infty} t_k(x_k)^* G y_k = 0.$$

With respect to the asymptotic behavior of the eigenvalues of matrix functions, we have the following result.

Lemma 9 ([5], Lemma 5). Let $F(t) = G(t) + t^{-1}H \in \mathbb{C}^{p \times p}$ where $G(t)$ is a Hermitian matrix function analytic on an open interval $J \subset \mathbb{R}$ around 0, and H is a constant Hermitian matrix such that $\text{rank}(H) = r$. Assume that H has a spectral decomposition

$$H = (V_1, V_2) \begin{pmatrix} \Lambda_r & O \\ O & O \end{pmatrix} (V_1, V_2)^*,$$

with unitary $V = (V_1, V_2)$ and $\Lambda_r \in \mathbb{R}^{r \times r}$ is a diagonal matrix. Then as t approaches 0, r eigenvalues of $F(t)$ tend in absolute value to ∞ , and the rest to the eigenvalues of $V_2^* G(0) V_2$.

Hence we will deduce the following result for singular values, which will be used in Section 4.

Lemma 10. Let $K(t) = L(t) + t^{-1}M \in \mathbb{C}^{p \times p}$, where $L(t)$ is an analytic matrix function on an open interval $J \subset \mathbb{R}$ around 0 and $\text{rank}(M) = s$. Consider the singular value decomposition of M

$$M = (P_1, P_2) \begin{pmatrix} \Sigma_s & O \\ O & O \end{pmatrix} (Q_1, Q_2)^*,$$

with unitary (P_1, P_2) and (Q_1, Q_2) and $\Sigma_s \in \mathbb{R}^{s \times s}$. Then as t approaches 0, s singular values of $K(t)$ tend to ∞ , and the rest to the singular values of the matrix $P_2^* L(0) Q_2$.

Proof. Observe that in the matrix function, valued in $\mathbb{C}^{2p \times 2p}$,

$$N(t) = \begin{pmatrix} O & K(t) \\ K^*(t) & O \end{pmatrix} = \begin{pmatrix} O & L(t) \\ L^*(t) & O \end{pmatrix} + t^{-1} \begin{pmatrix} O & M \\ M^* & O \end{pmatrix} = R(t) + t^{-1}S,$$

the matrices $R(t)$ and S are Hermitian, $R(t)$ is analytic around 0 and $\text{rank}(S) = 2s$. Let us note that, by the Jordan-Wielandt lemma, the eigenvalues of $N(t)$ are

$$\pm\sigma_1(K(t)), \dots, \pm\sigma_p(K(t)).$$

Consider the unitary matrix $(V_1, V_2) \in \mathbb{C}^{2p \times 2p}$, with

$$V_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & P_1 \\ Q_1 & -Q_1 \end{pmatrix} \in \mathbb{C}^{2p \times 2s}, \quad V_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} P_2 & P_2 \\ Q_2 & -Q_2 \end{pmatrix} \in \mathbb{C}^{2p \times 2(p-s)}.$$

Then

$$S = (V_1, V_2) \begin{pmatrix} \Sigma_s & O & O \\ O & -\Sigma_s & O \\ O & O & O \end{pmatrix} (V_1, V_2)^*,$$

is a spectral decomposition of S . Hence, by Lemma 9, when $t \rightarrow 0$, $2s$ eigenvalues of $N(t)$ tend in absolute value to ∞ ; and the rest to the eigenvalues of the matrix

$$V_2^* R(0) V_2 = \frac{1}{2} \begin{pmatrix} Q_2^* L^*(0) P_2 + P_2^* L(0) Q_2 & Q_2^* L^*(0) P_2 - P_2^* L(0) Q_2 \\ -Q_2^* L^*(0) P_2 + P_2^* L(0) Q_2 & -Q_2^* L^*(0) P_2 - P_2^* L(0) Q_2 \end{pmatrix}.$$

Taking the unitary matrix

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} -I & -I \\ -I & I \end{pmatrix},$$

we deduce that the eigenvalues of $V_2^* R(0) V_2$ are the eigenvalues of

$$X^* V_2^* R(0) V_2 X = \begin{pmatrix} O & P_2^* L(0) Q_2 \\ (P_2^* L(0) Q_2)^* & O \end{pmatrix},$$

that is, $\pm\sigma_1(P_2^* L(0) Q_2), \dots, \pm\sigma_{p-s}(P_2^* L(0) Q_2)$.

□

To conclude this section, we give some results about the singular values of matrix functions of a real variable. The first one can be seen in [6, Lemma 4 and (9)].

Lemma 11. *Let $F(t) \in \mathbb{C}^{q \times q}$ be an analytic matrix function on an open set $\Omega \subset \mathbb{R}$. Then, there exist unitary matrix functions $U(t), V(t)$ and a diagonal matrix function $\Sigma(t) = \text{diag}(\tilde{\sigma}_1(t), \tilde{\sigma}_2(t), \dots, \tilde{\sigma}_p(t)) \in \mathbb{R}^{q \times q}$, all of which are analytic on Ω , such that for $t \in \Omega$,*

$$U(t)^* F(t) V(t) = \Sigma(t).$$

Moreover

$$\tilde{\sigma}_i'(t) = \text{Re}(u_i^*(t) F'(t) v_i(t)).$$

Another result, which will be used in Section 5, is the following one [6, Lemma 5].

Lemma 12. *Let Ω be an open subset of \mathbb{R} and $F : \Omega \rightarrow \mathbb{C}^{m \times n}$ be an analytic matrix function on Ω . If the function $\sigma_i(F(t))$ has a positive local maximum (or minimum) at $t_0 \in \Omega$, then there exist a pair of singular vectors $u \in \mathbb{C}^{m \times 1}$, $v \in \mathbb{C}^{n \times 1}$ of $F(t_0)$ corresponding to $\sigma_i(F(t_0))$ such that*

$$\text{Re}(u^* F'(t_0) v) = 0.$$

4 Asymptotic behavior of the singular values

In this section, we analyze the asymptotic behavior of the singular values of the matrix function $S_2(t)$ defined in (19), both when $t \rightarrow 0^+$ and $t \rightarrow \infty$. We start with the $t \rightarrow 0^+$ case.

Lemma 13. *Let $S_2(t)$ be the matrix function in (19), and assume that $s = \text{rank}(C_1 B_1)$. Then when $t \rightarrow 0^+$, the s first singular values of $S_2(t)$ tend to ∞ , and the $2m - s$ remaining ones satisfy*

$$\lim_{t \rightarrow 0^+} \sigma_{s+k}(S_2(t)) = \sigma_k \begin{pmatrix} P_C L_1 (I_m - (C_1 B_1)^\dagger C_1 B_1) & O \\ O & (I_m - C_1 B_1 (C_1 B_1)^\dagger) L_1 P_B \end{pmatrix}.$$

for $k = 1, \dots, 2m - s$. If $\text{rank}(B_1) = \text{rank}(C_1) = n_1$, then when $t \rightarrow 0^+$, the n_1 first singular values of $S_2(t)$ tend to ∞ , and the $2m - n_1$ remaining ones satisfy

$$\lim_{t \rightarrow 0^+} \sigma_{n_1+k}(S_2(t)) = \sigma_k \begin{pmatrix} P_C L_1 P_B & O \\ O & P_C L_1 P_B \end{pmatrix} \text{ for } k = 1, \dots, 2m - n_1.$$

Remark 5. Note that the block $P_C L_1 P_B$ in the last matrix is repeated. This will matter later on.

Proof.

First, by (19), we have

$$S_2(t) = \begin{pmatrix} P_C L_1 & t P_C L_2 P_B \\ O & L_1 P_B \end{pmatrix} + t^{-1} \begin{pmatrix} O & O \\ -C_1 B_1 & O \end{pmatrix} = L(t) + t^{-1} M,$$

with $L(t)$ analytic in a neighborhood of 0 and $\text{rank}(M) = \text{rank}(C_1 B_1) = s$. Let $(U_1, U_2), (V_1, V_2)$ be unitary matrices of $\mathbb{C}^{m \times m}$, with $U_2, V_2 \in \mathbb{C}^{m \times (m-s)}$, such that

$$(U_1, U_2)^* (-C_1 B_1) (V_1, V_2) = \begin{pmatrix} \Sigma_s & O \\ O & O \end{pmatrix}, \quad (23)$$

with $\Sigma_s \in \mathbb{R}^{s \times s}$, gives us the singular value decomposition of $-C_1 B_1$. Therefore, considering the unitary matrices

$$P := \begin{pmatrix} O & O & I_m \\ U_1 & U_2 & O \end{pmatrix}, \quad Q := \begin{pmatrix} V_1 & V_2 & O \\ O & O & I_m \end{pmatrix},$$

we deduce that

$$P^* M Q = \begin{pmatrix} \Sigma_s & O \\ O & O \end{pmatrix}.$$

Calling

$$P_2 := \begin{pmatrix} O & I_m \\ U_2 & O \end{pmatrix}, \quad Q_2 := \begin{pmatrix} V_2 & O \\ O & I_m \end{pmatrix},$$

by Lemma 10 we see that when $t \rightarrow 0^+$, the s first singular values of $S_2(t)$ tend to ∞ , and the rest to the singular values of

$$P_2^* L(0) Q_2 = \begin{pmatrix} O & U_2^* L_1 P_B \\ P_C L_1 V_2 & O \end{pmatrix}.$$

Hence, for $k = 1, 2, \dots, 2m - s$,

$$\lim_{t \rightarrow 0^+} \sigma_{s+k}(S_2(t)) = \sigma_k \begin{pmatrix} P_C L_1 V_2 & O \\ O & U_2^* L_1 P_B \end{pmatrix}. \quad (24)$$

As by (23), $-C_1 B_1 V_1 = U_1 \Sigma_s$ and $-(C_1 B_1)^* U_1 = V_1 \Sigma_s$, from Lemma 6(1)(2) we get first

$$\begin{aligned} U_1 &= -C_1 B_1 V_1 \Sigma_s^{-1} \Rightarrow \text{Im}(U_1) \subset \text{Im}(C_1 B_1) = \text{Ker}(I_m - C_1 B_1 (C_1 B_1)^\dagger), \\ V_1 &= -(C_1 B_1)^* U_1 \Sigma_s^{-1} \Rightarrow \text{Im}(V_1) \subset \text{Im}((C_1 B_1)^*) = \text{Ker}(I_m - (C_1 B_1)^\dagger C_1 B_1). \end{aligned}$$

But, given that $I_m - C_1 B_1 (C_1 B_1)^\dagger$ and $I_m - (C_1 B_1)^\dagger C_1 B_1$ are orthogonal projectors in virtue of Lemma 3(1), we infer that

$$U_1^* (I_m - C_1 B_1 (C_1 B_1)^\dagger) = O, \quad (I_m - (C_1 B_1)^\dagger C_1 B_1) V_1 = O. \quad (25)$$

Similarly, from (23) and Lemma 6(1)(2), we see that

$$\begin{aligned} (C_1 B_1)^* U_2 = O &\Rightarrow \text{Im}(U_2) \subset \text{Ker}((C_1 B_1)^*) = \text{Im}(I_m - C_1 B_1 (C_1 B_1)^\dagger), \\ C_1 B_1 V_2 = O &\Rightarrow \text{Im}(V_2) \subset \text{Ker}(C_1 B_1) = \text{Im}(I_m - (C_1 B_1)^\dagger C_1 B_1). \end{aligned}$$

Thus

$$U_2^* (I_m - C_1 B_1 (C_1 B_1)^\dagger) = U_2^*, \quad (I_m - (C_1 B_1)^\dagger C_1 B_1) V_2 = V_2.$$

Consequently, with (25) and these two last equations, we deduce that

$$\begin{aligned} (O, P_C L_1 V_2) &= P_C L_1 (I_m - (C_1 B_1)^\dagger C_1 B_1) (V_1, V_2), \\ \begin{pmatrix} O \\ U_2^* L_1 P_B \end{pmatrix} &= \begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix} (I_m - C_1 B_1 (C_1 B_1)^\dagger) L_1 P_B. \end{aligned}$$

Substituting these equations in (24) we prove the lemma in the first case.

To prove the lemma in the $\text{rank}(B_1) = \text{rank}(C_1) = n_1$ case, as $s = \text{rank}(C_1 B_1) = n_1$, it is sufficient to see that $I_m - C_1 B_1 (C_1 B_1)^\dagger = P_C$, $I_m - (C_1 B_1)^\dagger C_1 B_1 = P_B$. In fact, given that $\text{rank}(B_1) = \text{rank}(C_1) = n_1$, then $C_1^\dagger C_1 = B_1 B_1^\dagger = I_{n_1}$ by Lemma 6(4). And $(C_1 B_1)^\dagger = B_1^\dagger C_1^\dagger$ by Lemma 7(4). Hence

$$\begin{aligned} I_m - C_1 B_1 (C_1 B_1)^\dagger &= I_m - C_1 B_1 B_1^\dagger C_1^\dagger = I_m - C_1 C_1^\dagger = P_C, \\ I_m - (C_1 B_1)^\dagger C_1 B_1 &= I_m - B_1^\dagger C_1^\dagger C_1 B_1 = I_m - B_1^\dagger B_1 = P_B. \end{aligned}$$

□

For the $t \rightarrow \infty$ case, we have the following result.

Lemma 14. *Let $S_2(t)$ be the matrix function in (19). Let us call $L := P_C L_2 P_B$, and assume that $\ell = \text{rank}(P_C L_2 P_B)$. Then when $t \rightarrow \infty$, the ℓ first singular values of $S_2(t)$ tend to ∞ , and the $2m - \ell$ remaining ones satisfy*

$$\lim_{t \rightarrow \infty} \sigma_{\ell+k}(S_2(t)) = \sigma_{\ell+k} \begin{pmatrix} P_C (I_m - L L^\dagger) L_1 & O \\ O & L_1 (I_m - L^\dagger L) P_B \end{pmatrix},$$

for $k = 1, \dots, 2m - \ell$.

Remark 6. Let us note that the matrix in the right hand side is $2m \times 2m$.

Proof.

Let $(U_1, U_2), (V_1, V_2)$ be unitary matrices of $\mathbb{C}^{m \times m}$, with $U_2, V_2 \in \mathbb{C}^{m \times (m-\ell)}$, that perform the singular value decomposition of L

$$(U_1, U_2)^* L (V_1, V_2) = \begin{pmatrix} \Sigma_\ell & O \\ O & O \end{pmatrix}, \quad (26)$$

with $\Sigma_\ell \in \mathbb{R}^{\ell \times \ell}$. Applying a similar reasoning to the one of the previous lemma for the matrix function

$$\hat{S}_2(t) = \begin{pmatrix} P_C L_1 & t^{-1} L \\ -t C_1 B_1 & L_1 P_B \end{pmatrix},$$

we find that when $t \rightarrow \infty$, the ℓ first singular values of $S_2(t)$ tend to ∞ , and the $2m - \ell$ remaining ones satisfy

$$\lim_{t \rightarrow \infty} \sigma_{\ell+k}(S_2(t)) = \sigma_k \begin{pmatrix} U_2^* P_C L_1 & O \\ O & L_1 P_B V_2 \end{pmatrix} \text{ for } k = 1, \dots, 2m - \ell. \quad (27)$$

Let us note that as P_C and P_B are orthogonal projectors, then $P_C L = L$ and $P_B L^* = L^*$. Hence, from (26) and by Lemma 6(1)(2), we obtain first

$$\begin{aligned} P_C U_1 \Sigma_\ell &= P_C L V_1 = L V_1 \Rightarrow \text{Im}(P_C U_1) \subset \text{Im} L = \text{Ker}(I_m - L L^\dagger), \\ P_B V_1 \Sigma_\ell &= P_B L^* U_1 = L^* U_1 \Rightarrow \text{Im}(P_B V_1) \subset \text{Im} L^* = \text{Ker}(I_m - L^\dagger L). \end{aligned}$$

Therefore, as by Lemma 3(1) $I_m - L L^\dagger$ and $I_m - L^\dagger L$ are orthogonal projectors, then

$$U_1^* P_C (I_m - L L^\dagger) = O, \quad (I_m - L^\dagger L) P_B V_1 = O. \quad (28)$$

Similarly, from (26) and Lemma 6(1)(2), we have

$$O = U_2^* L = U_2^* P_C L.$$

Thus

$$U_2^* P_C (I_m - L L^\dagger) = U_2^* P_C, \quad (I_m - L^\dagger L) P_B V_2 = P_B V_2.$$

Substituting these two last equalities and (28) in (27) we have proved the lemma. \square

Remark 7. Taking into account the expression for h given in (18), Lemmas 13 and 14, and Proposition 1, we conclude that if

$$2m + n_1 - 1 - \text{rank}(B_1) - \text{rank}(C_1) \leq \max\{\text{rank}(C_1 B_1), \text{rank}(P_C L_2 P_B)\}, \quad (29)$$

then $\sup_{t>0} \sigma_h(S_2(t)) = \infty$. That is, there is no matrix $X \in \mathbb{C}^{m \times m}$ such that 0 is a multiple eigenvalue of $M(\alpha, X)$. Consequently, Theorem 5 is proved in this case. Therefore, *from here on we will assume that*

$$2m + n_1 - 1 - \text{rank}(B_1) - \text{rank}(C_1) > \max\{\text{rank}(C_1 B_1), \text{rank}(P_C L_2 P_B)\}.$$

Remark 8. Once proved Theorem 5 we will be able to assert that

$$\min_{\substack{X \in \mathbb{C}^{m \times m} \\ \mathfrak{m}(0, M(\alpha, X)) \geq 2}} \|X - D\| = \infty$$

if and only if inequality (29) is satisfied.

In the next section the proof of Theorem 5 starts and continue until the end of Section 8.

5 When the supremum is a maximum

Given $t_0 \neq 0$, in agreement with the notations (18) and (19), let us call

$$\sigma_0 := \sigma_h(S_2(t_0)),$$

where we assume $\sigma_0 > 0$. Let

$$u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, v := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad (30)$$

be a pair of singular vectors of $S_2(t_0)$ associated with σ_0 , where $u_1, u_2, v_1, v_2 \in \mathbb{C}^{m \times 1}$.

Using [6, Section 4] and [3, Section 4] we will establish some properties of u, v . First, as $S_2(t_0)v = \sigma_0 u$ and $S_2(t_0)^*u = \sigma_0 v$, from (19) and (30) we get

$$P_C L_1 v_1 + t_0 P_C L_2 P_B v_2 = \sigma_0 u_1, \text{ and } t_0 P_B L_2^* P_C u_1 + P_B L_1^* u_2 = \sigma_0 v_2.$$

Hence, as $\sigma_0 > 0$, from the two previous equalities we deduce that $u_1 \in \text{Im}(P_C)$ and $v_2 \in \text{Im}(P_B)$. Thus, by Lemma 7(2)(3) we have $C_1^* u_1 = 0, P_C u_1 = u_1$ and $B_1 v_2 = 0, P_B v_2 = v_2$. These equalities joint with $S_2(t_0)v = \sigma_0 u$ and $S_2(t_0)^*u = \sigma_0 v$, imply the following equations.

$$P_C L_1 v_1 + t_0 P_C L_2 v_2 = \sigma_0 u_1, \quad (31)$$

$$-t_0^{-1} C_1 B_1 v_1 + L_1 v_2 = \sigma_0 u_2, \quad (32)$$

$$L_1^* u_1 - t_0^{-1} B_1^* C_1^* u_2 = \sigma_0 v_1, \quad (33)$$

$$t_0 P_B L_2^* u_1 + P_B L_1^* u_2 = \sigma_0 v_2, \quad (34)$$

$$C_1^* u_1 = C_1^\dagger u_1 = 0, \quad (35)$$

$$B_1 v_2 = 0, \quad (36)$$

$$P_C u_1 = u_1, \quad (37)$$

$$P_B v_2 = v_2. \quad (38)$$

Substituting (37) in (31) we see that $P_C(L_1 v_1 + t_0 L_2 v_2 - \sigma_0 u_1) = 0$. Therefore from Lemma 7(2) we have $L_1 v_1 + t_0 L_2 v_2 - \sigma_0 u_1 \in \text{Im}(C_1)$. Consequently by Lemma 6(3),

$$C_1 C_1^\dagger (L_1 v_1 + t_0 L_2 v_2 - \sigma_0 u_1) = L_1 v_1 + t_0 L_2 v_2 - \sigma_0 u_1. \quad (39)$$

Multiplying to the right equations (31)–(34) by $u_1^*, u_2^*, v_1^*, v_2^*$ respectively, conjugating (37) and (38), $u_1^* P_C = u_1^*, v_2^* P_B = v_2^*$, we conclude that

$$\begin{aligned} u_1^* L_1 v_1 + t_0 u_1^* L_2 v_2 &= \sigma_0 u_1^* u_1, \\ -t_0^{-1} u_2^* C_1 B_1 v_1 + u_2^* L_1 v_2 &= \sigma_0 u_2^* u_2, \\ v_1^* L_1^* u_1 - t_0^{-1} v_1^* B_1^* C_1^* u_2 &= \sigma_0 v_1^* v_1, \\ t_0 v_2^* L_2^* u_1 + v_2^* L_1^* u_2 &= \sigma_0 v_2^* v_2. \end{aligned}$$

Subtracting the conjugate of the third equation to the first one and the conjugate of the quarter equation to the second one, we conclude that

$$\sigma_0 (u_1^* u_1 - v_1^* v_1) = t_0 u_1^* L_2 v_2 + t_0^{-1} u_2^* C_1 B_1 v_1 = -\sigma_0 (u_2^* u_2 - v_2^* v_2). \quad (40)$$

Multiplying to the right (32) and (33) by u_1^* and v_2^* respectively and using $u_1^*C_1 = 0$ (35) and $B_1v_2 = 0$ (36), we obtain

$$u_1^*L_1v_2 = \sigma_0u_1^*u_2, \quad v_2^*L_1^*u_1 = \sigma_0v_2^*v_1.$$

Hence, subtracting the conjugate of the second equation to the first one, we see that $\sigma_0(u_1^*u_2 - v_1^*v_2) = 0$. As $\sigma_0 \neq 0$, we infer that

$$u_1^*u_2 = v_1^*v_2. \quad (41)$$

Remark 9. Note that e3 equations (31)–(41) remain valid for each pair of singular vectors associated with a nonzero singular value of $S_2(t)$ for $t \neq 0$. This remark will be important in Sections 6 and 8.

Now assume that $\sigma_h(S_2(t))$ attains a relative extremum $\sigma_0 := \sigma_h(S_2(t_0)) > 0$ at $t_0 \neq 0$. Then, by Lemma 12, there exists a pair of singular vectors u, v of $S_2(t_0)$ corresponding to $\sigma_h(S_2(t_0))$ such that

$$\operatorname{Re}(u^*S_2'(t_0)v) = \operatorname{Re}\left(u^* \begin{pmatrix} O & P_C L_2 P_B \\ t_0^{-2} C_1 B_1 & O \end{pmatrix} v\right) = 0.$$

Partitioning the vectors u, v according (30), we have

$$\operatorname{Re}(t_0^{-2}u_2^*C_1B_1v_1 + u_1^*P_C L_2 P_B v_2) = 0.$$

Since $t_0 \neq 0$ and $u_1^*P_C L_2 P_B v_2 = u_1^*v_2$ (by (37) and (38)), we deduce that $\operatorname{Re}(t_0^{-1}u_2^*C_1B_1v_1 + t_0u_1^*L_2v_2) = 0$. Hence, from (40), we see that

$$u_1^*u_1 = v_1^*v_1, \quad u_2^*u_2 = v_2^*v_2. \quad (42)$$

Now let us define the matrices

$$V := [v_1, v_2] \in \mathbb{C}^{m \times 2}, \quad U := [u_1, u_2] \in \mathbb{C}^{m \times 2}.$$

By (41) and (42), we have $V^*V = U^*U$. Hence, the matrix

$$D_0 := D - \sigma_0UV^\dagger,$$

satisfies $\|D - D_0\| = \sigma_0$ and

$$D_0V = DV - \sigma_0U, \quad U^*D_0 = U^*D - \sigma_0V^*. \quad (43)$$

(see [4], page 1208, (35)) Consequently, to prove Theorem 5 in this case, it suffices to prove that 0 is a multiple eigenvalue of the matrix $M(\alpha, D_0)$.

Since $\operatorname{rank}(V^*V) \geq 1$, we have two possibilities: $\operatorname{rank} V = 1$ or $\operatorname{rank} V = 2$. In the $\operatorname{rank} V = 1$ case, we will analyze the subcases when $v_2 \neq 0$ and when $v_2 = 0$.

5.1 $\operatorname{rank} V = 2$

Note that $\operatorname{rank} V = 2$ implies that v_1 and v_2 are linearly independent. Hence, to prove that 0 is a multiple eigenvalue of $M(\alpha, D_0)$ it suffices to see that

$$\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} \begin{pmatrix} z_2 & z_1 \\ w_2 & w_1 \\ v_2 & v_1 \end{pmatrix} = \begin{pmatrix} z_2 & z_1 \\ w_2 & w_1 \\ v_2 & v_1 \end{pmatrix} \begin{pmatrix} 0 & -t_0 \\ 0 & 0 \end{pmatrix},$$

with

$$\begin{aligned} z_2 &= -t_0^{-1}B_1v_1, & z_1 &= -C_1^\dagger(L_1v_1 + t_0L_2v_2 - \sigma_0u_1), \\ w_2 &= -\Sigma^{-1}B_2v_2, & w_1 &= t_0\Sigma^{-2}B_2v_2 - \Sigma^{-1}B_2v_1. \end{aligned}$$

Making operations and using that $B_1v_2 = 0$ (36) and $D_0v_2 = Dv_2 - \sigma_0u_2$ (43), the problem is reduced to verify the fulfillment of the equalities

$$\begin{aligned} -t_0^{-1}C_1B_1v_1 - C_2\Sigma^{-1}B_2v_2 + Dv_2 &= \sigma_0u_2, \\ -C_1C_1^\dagger(L_1v_1 + t_0L_2v_2 - \sigma_0u_1) + t_0C_2\Sigma^{-2}B_2v_2 - C_2\Sigma^{-1}B_2v_1 + Dv_1 - \sigma_0u_1 &= -t_0v_2. \end{aligned}$$

Since by (21) we have $L_1 = D - C_2\Sigma^{-1}B_2$ and $L_2 = I_m + C_2\Sigma^{-2}B_2$, the two previous equalities are reduced to

$$-t_0^{-1}C_1B_1v_1 + L_2v_2 = \sigma_0u_2, \quad -C_1C_1^\dagger(L_1v_1 + t_0L_2v_2 - \sigma_0u_1) + L_1v_1 + t_0L_2v_2 - \sigma_0u_1 = 0,$$

which are true by (32) and (39), respectively.

5.2 rank $V = 1$ and $v_2 \neq 0$

Observe that in this case $v_1 = \lambda v_2$ and $u_1 = \lambda u_2$, for some $\lambda \in \mathbb{C}$. Hence, as $v_2 \neq 0$, to prove that 0 is a multiple eigenvalue of $M(\alpha, D_0)$ it suffices to find a vector $w \in \mathbb{C}^{n_1 \times 1}$ such that

$$\begin{aligned} \begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} \begin{pmatrix} 0 & w \\ -\Sigma^{-1}B_2v_2 & -\Sigma^{-2}B_2v_2 \\ v_2 & 0 \end{pmatrix} \\ = \begin{pmatrix} 0 & w \\ -\Sigma^{-1}B_2v_2 & -\Sigma^{-2}B_2v_2 \\ v_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \end{aligned} \quad (44)$$

because this means that the columns of the matrix

$$\begin{pmatrix} 0 & w \\ -\Sigma^{-1}B_2v_2 & -\Sigma^{-2}B_2v_2 \\ v_2 & 0 \end{pmatrix}$$

form a Jordan chain of 0 as eigenvalue of $M(\alpha, D_0)$.

Multiplying the matrices in (44), we have

$$\begin{pmatrix} B_1v_2 & 0 \\ -B_2v_2 + B_2v_2 & -\Sigma^{-1}B_2v_2 \\ -C_2\Sigma^{-1}B_2v_2 + D_0v_2 & C_1w - C_2\Sigma^{-2}B_2v_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\Sigma^{-1}B_2v_2 \\ 0 & v_2 \end{pmatrix} \quad (45)$$

By (36) $B_1v_2 = 0$, so the (1,1)-entries in (45) are equal. By (43) $D_0v_2 = Dv_2 - \sigma_0u_2$, hence, using the definition of L_1 ,

$$-C_2\Sigma^{-1}B_2v_2 + D_0v_2 = Dv_2 - C_2\Sigma^{-1}B_2v_2 - \sigma_0u_2 = L_1v_2 - \sigma_0u_2.$$

As $v_1 = \lambda v_2$ and $B_1 v_2 = 0$, then $B_1 v_1 = 0$. From (32), $L_1 v_2 = \sigma_0 u_2$, thus the (3,1)–entries in (45) are equal. Equating the (3,2)–entries, and by the definition of L_2 , we have

$$C_1 w - C_2 \Sigma^{-2} B_2 v_2 = v_2, \quad C_1 w = v_2 + C_2 \Sigma^{-2} B_2 v_2, \quad C_1 w = L_2 v_2.$$

Thus the vector w must satisfy $C_1 w = L_2 v_2$. This vector exists if and only if $L_2 v_2 \in \text{Im } C_1 = \text{Ker } P_C$ by Lemma 7(2).

As $B_1 v_1 = 0$, $u_1 = \lambda u_2$, $v_1 = \lambda v_2$, from (31) and (37) we have

$$\lambda P_C L_1 v_2 + t_0 P_C L_2 v_2 = \lambda \sigma_0 u_2;$$

since $L_1 v_2 = \sigma_0 u_2$, we obtain $t_0 P_C L_2 v_2 = 0$. But $t_0 \neq 0$, so $P_C L_2 v_2 = 0$. Therefore, $L_2 v_2 \in \text{Ker } P_C$.

5.3 rank $V = 1$ and $v_2 = 0$

As $v_2 = 0$, then $u_2 = 0$, $v_1 \neq 0$ and $u_1 \neq 0$. Hence, to prove that 0 is a multiple eigenvalue of $M(\alpha, D_0)$ it suffices to find a vector $w \in \mathbb{C}^{n_1 \times 1}$, such that

$$\begin{pmatrix} 0 & -u_1^* C_2 \Sigma^{-1} & u_1^* \\ w^* & -u_1^* C_2 \Sigma^{-2} & 0 \end{pmatrix} \begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -u_1^* C_2 \Sigma^{-1} & u_1^* \\ w^* & -u_1^* C_2 \Sigma^{-2} & 0 \end{pmatrix}. \quad (46)$$

This means that the vectors

$$\begin{pmatrix} w \\ -\Sigma^{-2} C_2^* u_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\Sigma^{-1} C_2^* u_1 \\ u_1 \end{pmatrix}$$

form a Jordan chain on the left of 0 as an eigenvalue of $M(\alpha, D_0)$. Multiplying the matrices in (46), we will have to prove the following equality.

$$\begin{pmatrix} u_1^* C_1 & -u_1^* C_2 + u_1^* C_2 & -u_1^* C_2 \Sigma^{-1} B_2 + u_1^* D_0 \\ 0 & -u_1^* C_2 \Sigma^{-1} & w^* B_1 - u_1^* C_2 \Sigma^{-2} B_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -u_1^* C_2 \Sigma^{-1} & u_1^* \end{pmatrix}$$

The (1,1)–entries are equal, because $u_1^* C_1 = 0$ by (35). Let us see the reasons of the equality of the (1,3)–entries. By (43), $u_1^* D_0 = u_1^* D - \sigma_0 v_1^*$. So,

$$-u_1^* C_2 \Sigma^{-1} B_2 + u_1^* D_0 = 0 \iff -u_1^* C_2 \Sigma^{-1} B_2 + u_1^* D = \sigma_0 v_1^*.$$

By the definition of L_1 , the last equality is equivalent to $u_1^* L_1 = \sigma_0 v_1^*$. As $u_2 = 0$, (33) implies $L_1^* u_1 = \sigma_0 v_1^*$. Finally to prove the equality of the (2,3)–entries, we must prove the existence of a vector w such that

$$w^* B_1 - u_1^* C_2 \Sigma^{-2} B_2 = u_1^*.$$

By the definition of L_2 , the vector w must satisfy $w^* B_1 = u_1^* L_2$; that is $B_1^* w = L_2^* u_1$. Such a w exists if and only if $L_2^* u_1 \in \text{Im } B_1^* = \text{Ker } P_B$, by Lemma 7(3). Since $u_2 = v_2 = 0$, $t_0 \neq 0$, and (34), $P_B L_2^* u_1 = 0$.

Remark 10. We have proved Theorem 5 when the function $t \mapsto \sigma_h(S_2(t))$ has a positive local extremum at a point $t_0 \neq 0$. Note that if for a positive integer q we have $\sigma_{h+q}(S_2(t)) \neq 0$, for $t \neq 0$, we can apply the same reasoning to the function $t \mapsto \sigma_{h+q}(S_2(t))$. Therefore, as in [4, Corollary 30], we deduce the following result.

Corollary 15. *The function $t \mapsto \sigma_h(S_2(t))$ has no relative minimum in $(0, \infty)$. Moreover for each positive integer q , either $\sigma_{h+q}(S_2(t)) = 0$ for $t \neq 0$, or the function $t \mapsto \sigma_{h+q}(S_2(t))$ has no relative minimum in $(0, \infty)$.*

6 When the supremum is the limit at ∞

In this section we study the case when

$$\sup_{t>0} \sigma_h(S_2(t)) = \lim_{t \rightarrow \infty} \sigma_h(S_2(t)) = \sigma_0.$$

Observe first that Lemma 14 warrants the existence of the limit, which is finite because $h > \text{rank}(P_C L_2 P_B)$. Consider now a sequence of real numbers $\{t_k\}_{k=1}^\infty$ which tends to ∞ when $k \rightarrow \infty$, and let denote $\hat{\sigma}_k := \sigma_p(S_2(t_k))$. Then

$$\lim_{k \rightarrow \infty} \hat{\sigma}_k = \sigma_0.$$

For each k , let

$$u^k := \begin{pmatrix} u_1^k \\ u_2^k \end{pmatrix}, \quad v^k := \begin{pmatrix} v_1^k \\ v_2^k \end{pmatrix}, \quad u_i^k, v_i^k \in \mathbb{C}^{m \times 1}, \quad i = 1, 2,$$

be pairs of singular vectors of $S_2(t_k)$, associated with $\hat{\sigma}_k$. As the vectors u^k and v^k are unitary, the sequence $\{(u^k, v^k)\}_{k=1}^\infty$ has a convergent subsequence, say to (u, v) . In order to simplify we will denote the terms of this subsequence with the same index k . Then

$$\lim_{k \rightarrow \infty} u^k = u =: \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \lim_{k \rightarrow \infty} v^k = v =: \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

For each sufficiently large k , the equalities (31)–(38), (40), and (41) are satisfied for t_k, u^k, v^k and $\hat{\sigma}_k$ instead of t_0, u, v and σ_0 . Hence, taking limits, we infer that

$$\lim_{k \rightarrow \infty} P_C L_2 v_2^k = P_C L_2 v_2 = 0, \quad (47)$$

$$L_1 v_2 = \sigma_0 u_2, \quad (48)$$

$$L_1^* u_1 = \sigma_0 v_1, \quad (49)$$

$$\lim_{k \rightarrow \infty} t_k P_B L_2^* u_1^k = \sigma_0 v_2 - P_B L_1^* u_2, \quad (50)$$

$$\lim_{k \rightarrow \infty} P_B L_2^* u_1^k = P_B L_2^* u_1 = 0, \quad (51)$$

$$C_1^* u_1 = C_1^\dagger u_1 = 0, \quad (52)$$

$$B_1 v_2 = 0, \quad (53)$$

$$\lim_{k \rightarrow \infty} t_k (u_1^k)^* L_2 v_2^k = \sigma_0 (u_1^* u_1 - v_1^* v_1) = -\sigma_0 (u_2^* u_2 - v_2^* v_2), \quad (54)$$

$$u_1^* u_2 = v_1^* v_2. \quad (55)$$

We are going to apply Lemma 8 to $t_k(u_1^k)^*L_2v_2^k = t_k(u_1^k)^*P_C L_2 P_B v_2^k$, for each k , because $(u_1^k)^* = (u_1^k)^*P_C$ and $v_2^k = P_B v_2^k$, by (37) and (38), respectively. Let $x_k := u_1^k$, $y_k := v_2^k$ and $G := P_C L_2 P_B$. Then by (47) we have

$$\lim_{k \rightarrow \infty} G y_k = \lim_{k \rightarrow \infty} P_C L_2 P_B v_2^k = \lim_{k \rightarrow \infty} P_C L_2 v_2^k = 0.$$

On the other hand, $\|t_k x_k^* G\| = \|t_k (u_1^k)^* P_C L_2 P_B\| = \|t_k P_B L_2^* u_1^k\|$ is bounded in virtue of (50). Thus, applying Lemma 8, we see that

$$\lim_{k \rightarrow \infty} t_k (u_1^k)^* L_2 v_2^k = \lim_{k \rightarrow \infty} t_k (u_1^k)^* P_C L_2 P_B v_2^k = 0.$$

Substituting this equality in (54), we conclude that $u_1^* u_1 = v_1^* v_1$ and $u_2^* u_2 = v_2^* v_2$. Hence, if we consider the matrices $V := [v_1, v_2]$, $U := [u_1, u_2]$, from the two preceding equalities and (55), we have $U^* U = V^* V$. Therefore, as in Section 5, the matrix

$$D_0 := D - \sigma_0 U V^\dagger,$$

satisfies $\|D - D_0\| = \sigma_0$ and

$$D_0 v_2 = D v_2 - \sigma_0 u_2, \quad u_1^* D_0 = u_1^* D - \sigma_0 v_1^*.$$

By the definition of L_1 , given in (21), from (48) and (49), we see that

$$D_0 v_2 = C_2 \Sigma^{-1} B_2 v_2, \quad u_1^* D_0 = u_1^* C_2 \Sigma^{-1} B_2. \quad (56)$$

Hence, to prove Theorem 5 in this case, it suffices to prove that 0 is a multiple eigenvalue of the matrix $M(\alpha, D_0)$. Once here, we are going to consider two cases: $v_2 \neq 0$ and $v_2 = 0$.

6.1 $v_2 \neq 0$

As v_2 is nonzero, to prove that 0 is a multiple eigenvalue of the matrix $M(\alpha, D_0)$, it suffices to find a vector $w \in \mathbb{C}^{n_1 \times 1}$ such that

$$\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} \begin{pmatrix} 0 & w \\ -\Sigma^{-1} B_2 v_2 & -\Sigma^{-2} B_2 v_2 \\ v_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & w \\ -\Sigma^{-1} B_2 v_2 & -\Sigma^{-2} B_2 v_2 \\ v_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Multiplying these matrices, as $B_1 v_2 = 0$ by (53), and $D_0 v_2 = C_2 \Sigma^{-1} B_2 v_2$ by (56), the problem is reduced to find a vector w that satisfies $C_1 w - C_2 \Sigma^{-2} B_2 v_2 = v_2$. That is, using the definition of L_2 , given in (21), it suffices to find w such that

$$C_1 w = L_2 v_2.$$

Hence, there exists w if and only if $L_2 v_2 \in \text{Im } C_1$, or which is equivalent by Lemma 7(2), if and only if $L_2 v_2 \in \text{Ker } P_C$, which is true by (47).

6.2 $v_2 = 0$

In this case $u_2 = 0$ and $u_1 \neq 0$. Thus, it suffices to find a vector $w \in \mathbb{C}^{n_1 \times 1}$ such that

$$\begin{pmatrix} 0 & -u_1^* C_2 \Sigma^{-1} & u_1^* \\ w^* & -u_1^* C_2 \Sigma^{-2} & 0 \end{pmatrix} \begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -u_1^* C_2 \Sigma^{-1} & u_1^* \\ w^* & -u_1^* C_2 \Sigma^{-2} & 0 \end{pmatrix}.$$

Multiplying these matrices, as $u_1^*C_1 = 0$ by (52) and $u_1^*D_0 = u_1^*C_2\Sigma^{-1}B_2$ by (56), it suffices to find a vector w such that

$$w_1^*B_1 - u_1^*C_2\Sigma^{-2}B_2 = u_1^* \Leftrightarrow B_1^*w_1 = L_2^*u_1,$$

having used the definition of L_2 , given in (21). Consequently, there exists w if and only if $L_2^*u_1 \in \text{Im } B_1$, or which is equivalent by Lemma 7(3), if and only if $L_2^*u_1 \in \text{Ker } P_B$; which is true by (51).

7 When the supremum is the limit at 0, and $\text{rank}(B_1) < n_1$ or $\text{rank}(C_1) < n_1$

In this section we assume that $\text{rank}(B_1) < n_1$ or $\text{rank}(C_1) < n_1$, and we consider the case when

$$\sup_{t>0} \sigma_h(S_2(t)) = \lim_{t \rightarrow 0^+} \sigma_h(S_2(t)) = \sigma_0.$$

To shorten notation, we write s instead of $\text{rank}(C_1B_1)$. First, let us observe that Lemma 13 warrants the existence of the limit. Moreover, by the same lemma and denoting

$$T_1 := I_m - (C_1B_1)^\dagger C_1B_1, \quad T_2 := I_m - C_1B_1(C_1B_1)^\dagger,$$

as $h > s$, we have

$$\lim_{t \rightarrow 0^+} \sigma_h(S_2(t_0)) = \sigma_{h-s} \begin{pmatrix} P_C L_1 T_1 & O \\ O & T_2 L_1 P_B \end{pmatrix} = \sigma_0 > 0.$$

We are going to prove some properties of the singular vectors of $P_C L_1 T_1$ and $T_2 L_1 P_B$. Assume that σ_0 is a singular value of $P_C L_1 T_1$ and let (u, v) be a pair of singular vectors corresponding to it. As $P_C L_1 T_1 v = \sigma_0 u$, by Lemma 7(2), we have $u \in \text{Im}(P_C) = \text{Ker}(C_1^*)$, that is $P_C u = u$, $u^* C_1 = 0$. On the other hand, as $T_1 L_1^* P_C u = T_1 L_1^* u = \sigma_0 v$,

$$L_1^* u - \sigma_0 v = (C_1 B_1)^\dagger C_1 B_1 L_1^* u \in \text{Im}(C_1 B_1)^\dagger = \text{Im}(C_1 B_1)^* \subset \text{Im}(B_1^*).$$

Hence, by Lemma 6(3), we see that $u^* L_1 - \sigma_0 v^* = (u^* L_1 - \sigma_0 v^*) B_1^\dagger B_1$. Thus, as (u, v) is a pair singular vectors of $P_C L_1 T_1$ associated with σ_0 , we infer that

$$P_C L_1 T_1 v = \sigma_0 u, \tag{57}$$

$$T_1 L_1^* u = \sigma_0 v, \tag{58}$$

$$P_C u = u, \quad u^* C_1 = 0, \tag{59}$$

$$u^* L_1 - \sigma_0 v^* = (u^* L_1 - \sigma_0 v^*) B_1^\dagger B_1. \tag{60}$$

Similarly, using Lemmas 7(3) and 6(3), if (x, y) is a pair of singular vectors of $T_2 L_1 P_B$ associated with σ_0 , we conclude that

$$T_2 L_1 y = \sigma_0 x, \tag{61}$$

$$P_B L_1^* T_2 x = \sigma_0 y, \tag{62}$$

$$P_B y = y, \quad B_1 y = 0, \tag{63}$$

$$L_1 y - \sigma_0 x = C_1 C_1^\dagger (L_1 y - \sigma_0 x). \tag{64}$$

To concluding the proof of Theorem 5 in this case, we are going to consider two cases: (1) σ_0 is a singular value of $P_C L_1 T_1$; (2) σ_0 is a singular value of $T_2 L_1 P_B$.

7.1 σ_0 is a singular value of $P_C L_1 T_1$

Let (u, v) be a pair of singular vectors of $P_C L_1 T_1$ associated with σ_0 . For the entire subsection let

$$D_0 := D - \sigma_0 u v^*.$$

It is clear that $\|D - D_0\| = \sigma_0$. Besides, by (60), we have

$$\left(-(u^* L_1 - \sigma_0 v^*) B_1^\dagger, -u^* C_2 \Sigma^{-1}, u^* \right) \begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} = 0. \quad (65)$$

At this point, we consider two subcases: (1) $\text{rank}(B_1) < n_1$ and (2) $\text{rank}(C_1) < n_1 = \text{rank}(B_1)$.

7.1.1 $\text{rank}(B_1) < n_1$

In this case, there exists a nonzero vector $z \in \mathbb{C}^{n_1 \times 1}$ such that $z^* B_1 = 0$. Thus,

$$(z^*, 0, 0) \begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} = 0.$$

This, together with (65), proves that 0 is a multiple eigenvalue of $M(\alpha, D_0)$.

7.1.2 $\text{rank}(C_1) < n_1 = \text{rank}(B_1)$

As $\text{rank}(C_1) < n_1$ there exists a nonzero vector $z \in \mathbb{C}^{n_1 \times 1}$ such that $C_1 z = 0$. Therefore

$$\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix} = 0.$$

Thus, by (65), to prove that 0 is a multiple eigenvalue of $M(\alpha, D_0)$ it suffices to see that

$$\left(-(u^* L_1 - \sigma_0 v^*) B_1^\dagger, -u^* C_2 \Sigma^{-1}, u^* \right) \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix} = -(u^* L_1 - \sigma_0 v^*) B_1^\dagger z = 0.$$

Since $\text{rank}(B_1) = n_1$, (58) implies $L_1^* u - \sigma_0 v = (C_1 B_1)^\dagger C_1 B_1 L_1^* u$ and $B_1 B_1^\dagger = I_{n_1}$. Thus

$$(u^* L_1 - \sigma_0 v^*) B_1^\dagger z = u^* L_1 (C_1 B_1)^\dagger C_1 B_1 B_1^\dagger z = u^* L_1 (C_1 B_1)^\dagger C_1 z = 0,$$

because $C_1 z = 0$.

7.2 σ_0 is a singular value of $T_2 L_1 P_B$

Let (x, y) be a pair of singular vectors of $T_2 L_1 P_B$ associated with σ_0 . In this subsection we define

$$D_0 := D - \sigma_0 x y^*.$$

Again we have $\|D - D_0\| = \sigma_0$. From (64), we see that

$$\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} \begin{pmatrix} -C_1^\dagger(L_1 y - \sigma_0 x) \\ -\Sigma^{-1} B_2 y \\ y \end{pmatrix} = 0. \quad (66)$$

Now we will consider two subcases: (1) $\text{rank}(C_1) < n_1$ and (2) $\text{rank}(B_1) < n_1 = \text{rank}(C_1)$.

7.2.1 $\text{rank}(C_1) < n_1$

In this case there exists a nonzero vector $z \in \mathbb{C}^{n_1 \times 1}$ such that $C_1 y = 0$. Hence,

$$\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix} = 0.$$

This, together with (66) proves that 0 is a multiple eigenvalue of $M(\alpha, D_0)$.

7.2.2 $\text{rank}(B_1) < n_1 = \text{rank}(C_1)$

As $\text{rank}(B_1) < n_1$ there is a nonzero vector $z \in \mathbb{C}^{n_1 \times 1}$ such that $z^* B_1 = 0$. So

$$(z^*, 0, 0) \begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} = 0.$$

Therefore, to demonstrate that 0 is a multiple eigenvalue of $M(\alpha, D_0)$, it suffices to see that

$$(z^*, 0, 0) \begin{pmatrix} -C_1^\dagger(L_1 y - \sigma_0 x) \\ -\Sigma^{-1} B_2 y \\ y \end{pmatrix} = -z^* C_1^\dagger(L_1 y - \sigma_0 x) = 0.$$

Since $\text{rank}(C_1) = n_1$, (61) implies $L_1 y - \sigma_0 x = C_1 B_1 (C_1 B_1)^\dagger L_1 y$ and $C_1^\dagger C_1 = I_{n_1}$. Consequently

$$z^* C_1^\dagger(L_1 y - \sigma_0 x) = z^* C_1^\dagger C_1 B_1 (C_1 B_1)^\dagger L_1 y = z^* B_1 (C_1 B_1)^\dagger L_1 y = 0,$$

because $z^* B_1 = 0$.

8 When the supremum is the limit at 0, and $\text{rank}(B_1) = \text{rank}(C_1) = n_1$

In this section we assume that $\text{rank}(B_1) = \text{rank}(C_1) = n_1$, and we consider the case when

$$\sup_{t>0} \sigma_h(S_2(t)) = \lim_{t \rightarrow 0^+} \sigma_h(S_2(t)).$$

As $\text{rank}(B_1) = \text{rank}(C_1) = n_1$, by Lemma 6(5), we have $C_1^\dagger C_1 = B_1 B_1^\dagger = I_{n_1}$; this fact will be used frequently along the section. Besides from (18) it follows that $h = 2m - n_1 - 1$. Since $\text{rank}(C_1 B_1) = n_1$, by Lemma 13 we have

$$\lim_{t \rightarrow 0^+} \sigma_h(S_2(t)) = \lim_{t \rightarrow 0^+} \sigma_{h+1}(S_2(t)) = \sigma_{m-n_1}(P_C L_1 P_B) =: \sigma_0 > 0.$$

Thus there exists an $\varepsilon > 0$ such that the functions $t \mapsto \sigma_h(S_2(t))$ and $t \mapsto \sigma_{h+1}(S_2(t))$ are nonincreasing on the interval $(0, \varepsilon)$.

If σ_0 is a multiple singular value of $P_C L_1 P_B$, then there are pairs of singular vectors $(u_1, v_1), (u_2, v_2)$ of $P_C L_1 P_B$ associated with σ_0 so that $U^* U = I_2 = V^* V$, where $U := [u_1, u_2]$ and $V := [v_1, v_2]$. Define now the matrix

$$D_0 := D - \sigma_0 U V^*.$$

Since $\|U V^*\| = 1$, it follows that $\|D - D_0\| = \sigma_0$ and $U^* D_0 = U^* D - \sigma_0 V^*$. Given that $L_1 = D - C_2 \Sigma^{-1} B_2$, by (59) and (60) we have

$$(-(u_i^* L_1 - \sigma_0 v_i^*) B_1^\dagger, -u_i^* C_2 \Sigma^{-1}, u_i^*) \begin{pmatrix} 0 & 0 & B_1 \\ 0 & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} = 0, \quad i = 1, 2;$$

that is, 0 is a multiple eigenvalue of $M(\alpha, D_0)$.

Now assume σ_0 is a simple singular value of $P_C L_1 P_B$. We will consider the matrix function $t \mapsto t S_2(t)$, which is analytic on \mathbb{R} . Then, by Lemma 11, there must be some $2m \times 2m$ unitary matrix functions

$$U(t) := (U_1(t), U_2(t) \dots U_{2m}(t)), \quad V(t) := (V_1(t), V_2(t) \dots V_{2m}(t))$$

and a diagonal matrix function $\Sigma(t) = \text{diag}(\tilde{\sigma}_1(t), \tilde{\sigma}_2(t), \dots, \tilde{\sigma}_{2m}(t)) \in \mathbb{R}^{2m \times 2m}$, all analytic on \mathbb{R} , so that for each $t \neq 0$ we have

$$U(t)^* t S_2(t) V(t) = \Sigma(t) \Leftrightarrow U(t)^* S_2(t) V(t) = \text{diag}(\tilde{\sigma}_i(t)/t).$$

Observe that for some interval $(0, a)$, with $a > 0$, we can assume without loss of generality that all the functions $\tilde{\sigma}_i(t)$ are nonnegative on it. Let j, k be now the unique subscripts such that

$$\lim_{t \rightarrow 0^+} \frac{\tilde{\sigma}_j(t)}{t} = \lim_{t \rightarrow 0^+} \frac{\tilde{\sigma}_k(t)}{t} = \sigma_0.$$

Thus, it is correct to assume that for each positive t sufficiently close to 0 we have $\tilde{\sigma}_j(t) \geq \tilde{\sigma}_k(t)$. Define the functions

$$f(t) := \frac{\tilde{\sigma}_j(t)}{t}, \quad g(t) := \frac{\tilde{\sigma}_k(t)}{t}.$$

Then, we see that

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{t \rightarrow 0^+} g(t) = \sigma_0,$$

and there exists a $b > 0$ such that $f(t), g(t)$ are analytic on $(0, b)$, and for $t \in (0, b)$ we have the inequality $f(t) \geq g(t)$.

Let us denote

$$u(t) := \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = U_j(t), \quad v(t) := \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = V_j(t),$$

and

$$x(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = U_k(t), \quad y(t) := \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = V_k(t),$$

where $U_j(t), U_k(t)$ and $V_j(t), V_k(t)$ are the column vectors of $U(t)$ and $V(t)$ corresponding to the places j and k , respectively. Since they are analytic functions, we infer that the following limits exist

$$\lim_{t \rightarrow 0^+} u(t) = u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \lim_{t \rightarrow 0^+} v(t) = v := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

and

$$\lim_{t \rightarrow 0^+} x(t) = x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \lim_{t \rightarrow 0^+} y(t) = y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Moreover, $(u(t), v(t))$ and $(x(t), y(t))$ are pairs of singular vectors of $S_2(t)$ associated with the singular values $f(t)$ and $g(t)$, respectively. Therefore, for each $t \in (0, b)$ the equalities (31)–(41) for $(u(t), v(t))$ and $f(t)$, and for $(x(t), y(t))$ and $g(t)$, respectively, are satisfied.

First note that from (32) we deduce that

$$\lim_{t \rightarrow 0^+} C_1 B_1 v_1(t) = C_1 B_1 v_1 = 0.$$

Thus, as $\text{rank}(C_1) = n_1$, we have $B_1 v_1 = 0$, which is equivalent to $P_B v_1 = v_1$ by Lemma 7(3). Similarly, as $\text{rank}(B_1) = n_1$, being aware of Remark 9 and taking limits in (33) when $t \rightarrow 0^+$, we conclude that $C_1^* u_2 = 0$, which is equivalent to $P_C u_2 = u_2$ by Lemma 7(2).

Now, being aware of Remark 9 and considering equations (31)–(38), changing t_0 by t in them and as $\text{Im}(C_1) = \text{Ker}(P_C)$, $\text{Im}(B_1^*) = \text{Ker}(P_B)$, by Lemma 7(2)(3), when $t \rightarrow 0^+$ we infer that

$$P_C L_1 v_1 = \sigma_0 u_1, \tag{67}$$

$$\lim_{t \rightarrow 0^+} t^{-1} C_1 B_1 v_1(t) = L_1 v_2 - \sigma_0 u_2, \tag{68}$$

$$L_1 v_2 - \sigma_0 u_2 \in \text{Im}(C_1) = \text{Ker}(P_C), \tag{69}$$

$$\lim_{t \rightarrow 0^+} t^{-1} B_1^* C_1^* u_2(t) = L_1^* u_1 - \sigma_0 v_1, \tag{70}$$

$$L_1^* u_1 - \sigma_0 v_1 \in \text{Im}(B_1^*) = \text{Ker}(P_B), \tag{71}$$

$$P_B L_1^* u_2 = \sigma_0 v_2, \tag{72}$$

$$C_1^* u_1 = C_1^* u_2 = 0, \tag{73}$$

$$B_1 v_1 = B_1 v_2 = 0, \tag{74}$$

$$P_C u_1 = u_1, P_C u_2 = u_2, \tag{75}$$

$$P_B v_1 = v_1, P_B v_2 = v_2. \tag{76}$$

Let us see that

$$P_C L_1 P_B \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sigma_0 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad P_B L_1^* P_C \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \sigma_0 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{77}$$

First note that, as $P_B v_i = v_i$ and $P_C u_i = u_i, i = 1, 2$, the demonstration of (77) is equivalent to the proof of

$$P_C L_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sigma_0 \begin{pmatrix} u_1 \\ P_C u_2 \end{pmatrix}, \quad P_B L_1^* \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \sigma_0 \begin{pmatrix} P_B v_1 \\ v_2 \end{pmatrix}.$$

But

$$P_C L_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sigma_0 \begin{pmatrix} u_1 \\ P_C u_2 \end{pmatrix} \Leftrightarrow \begin{cases} P_C L_1 v_1 = \sigma_0 u_1, \\ P_C (L_1 v_2 - \sigma_0 u_2) = 0, \end{cases}$$

these last equalities are true by (67) and (69), respectively. Secondly, note that

$$P_B L_1^* \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \sigma_0 \begin{pmatrix} P_B v_1 \\ v_2 \end{pmatrix} \Leftrightarrow \begin{cases} P_B (L_1^* u_1 - \sigma_0 v_1) = 0, \\ P_B L_1^* u_2 = \sigma_0 v_2, \end{cases}$$

true by (71) and (72), respectively.

On the other hand, being aware of Remark 9 and taking limits in (40) we obtain

$$\lim_{t \rightarrow 0^+} t^{-1} u_2(t)^* C_1 B_1 v_1(t) = \sigma_0 (u_1^* u_1 - v_1^* v_1) = -\sigma_0 (u_2^* u_2 - v_2^* v_2).$$

Besides, by (68) and (70), we have $C_1 B_1 v_1(t) \rightarrow 0$ and $t^{-1} B_1^* C_1^* u_2(t) \rightarrow L_1^* u_1 - \sigma_0 v_1$. Thus, by Lemma 8, we deduce that

$$\lim_{t \rightarrow 0^+} t^{-1} u_2(t)^* C_1 B_1 v_1(t) = 0.$$

Therefore

$$u_1^* u_1 - v_1^* v_1 = u_2^* u_2 - v_2^* v_2 = 0. \quad (78)$$

Too, being aware of Remark 9 and taking limits in (41), we obtain

$$u_1^* u_2 = v_1^* v_2.$$

Hence, if we consider the matrices $V := [v_1, v_2], U := [u_1, u_2] \in \mathbb{C}^{m \times 2}$, from (78) and the last equality, we find that $U^* U = V^* V$. Thus, as in Section 5, the matrix

$$D_0 := D - \sigma_0 U V^\dagger,$$

satisfy $\|D - D_0\| = \sigma_0$ and $D_0 V = D V - \sigma_0 U$.

Remark that all the above properties are true also for (x, y) and $X := [x_1, x_2], Y := [y_1, y_2]$.

So, to prove Theorem 5 in this case, it suffices to prove that 0 is a multiple eigenvalue of the matrix $M(\alpha, D_0)$, where $D_0 := D - \sigma_0 U V^\dagger$ or $D_0 := D - \sigma_0 X Y^\dagger$, respectively. The following lemma allows us to reduce the possible cases.

Lemma 16. *With the preceding notations, we have*

- (1) $\text{rank}(U) = \text{rank}(V) = \text{rank}(X) = \text{rank}(Y) = 1$,
- (2) if $v_1 = 0$ then $y_2 = 0$,
- (3) if $v_2 = 0$ then $y_1 = 0$.

Proof.

(1) Let us suppose that $\text{rank}(U) = \text{rank}(V) = 2$, that is, both $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are linearly independent. Hence, as $\|u_1\| = \|v_1\|$ and $\|u_2\| = \|v_2\|$ by (78), from (77) we see that

$$(\tilde{u}_1, \tilde{v}_1) = \left(\frac{u_1}{\|v_1\|}, \frac{v_1}{\|v_1\|} \right) \quad \text{and} \quad (\tilde{u}_2, \tilde{v}_2) = \left(\frac{u_2}{\|v_2\|}, \frac{v_2}{\|v_2\|} \right),$$

are pairs of singular vectors of $P_C L_1 P_B$ associated with the simple singular value σ_0 . Therefore, $(\tilde{u}_1, \tilde{u}_2)$ as well as $(\tilde{v}_1, \tilde{v}_2)$ would be linearly dependent. A contradiction.

(2) Assume now that $v_1 = 0$, hence $v_2 \neq 0$. Then, as u, y are orthogonal, we have $v_2^* y_2 = 0$. Let us suppose that $y_2 \neq 0$. Repeating the previous reasoning, it follows that $y_2/\|y_2\|$ and $v_2/\|v_2\|$ right singular vectors of $P_C L_1 P_B$ associated with σ_0 . As σ_0 is simple, then both vectors are linearly dependent and nonzero. The same happens for v_2, y_2 . Consequently it is not possible that $v_2^* y_2 = 0$. In a similar way is proved (3). □

At this moment, by the preceding lemma, the possible cases to analyze are two: (1) $v_1 = 0$ or $v_2 = 0$; (2) $u_1 = \alpha u_2$, $v_1 = \alpha v_2$, $x_1 = \beta x_2$ and $y_1 = \beta y_2$, with scalar nonzero α, β .

8.1 $v_1 = 0$ or $v_2 = 0$

First let us suppose that $v_1 = 0$ and let $D_0 := D - \sigma_0 U V^\dagger$. Note that $u_1 = 0$, hence v_2 and u_2 are nonzero vectors. To prove Theorem 5 in this case, we will search a pair of eigenvectors of $M(\alpha, D_0)$ associated with the eigenvalue 0, one on the left and other on the right, so that they are orthogonal.

We are going to prove that

$$\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} \begin{pmatrix} -C_1^\dagger(L_1 v_2 - \sigma_0 u_2) \\ -\Sigma^{-1} B_2 v_2 \\ v_2 \end{pmatrix} = 0.$$

Multiplying, as $B_1 v_2 = 0$ and $D_0 v_2 = D v_2 - \sigma_0 u_2$, just check us that

$$-C_1 C_1^\dagger(L_1 v_2 - \sigma_0 u_2) - C_2 \Sigma^{-1} B_2 v_2 + D v_2 - \sigma_0 u_2 = 0.$$

Or which is the same,

$$C_1 C_1^\dagger(L_1 v_2 - \sigma_0 u_2) = L_1 v_2 - \sigma_0 u_2,$$

because by (21), $L_1 = D - C_2 \Sigma^{-1} B_2$. That is, by Lemma 6(3), it suffices to prove that $L_1 v_2 - \sigma_0 u_2 \in \text{Im } C_1$. Which is true by (69).

On the other hand, since $P_B v_2 = v_2$, from (72) we conclude that $L_1^* u_2 - \sigma_0 v_2 \in \text{Ker}(P_B) = \text{Im}(B_1^\dagger)$. Hence, reasoning in a similar manner, it follows that

$$(-(u_2^* L_1 - \sigma_0 v_2^*) B_1^\dagger, -u_2^* C_2 \Sigma^{-1}, u_2^*) \begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} = 0.$$

By the definition (21), $L_2 = I_m + C_2 \Sigma^{-2} B_2$. Moreover $v_2^* B_1^\dagger = 0$, by (74). Let us denote by ϕ the following scalar:

$$\begin{aligned} \phi &:= (-(u_2^* L_1 - \sigma_0 v_2^*) B_1^\dagger, -u_2^* C_2 \Sigma^{-1}, u_2^*) \begin{pmatrix} -C_1^\dagger(L_1 v_2 - \sigma_0 u_2) \\ -\Sigma^{-1} B_2 v_2 \\ v_2 \end{pmatrix} \\ &= u_2^* L_1 B_1^\dagger C_1^\dagger(L_1 v_2 - \sigma_0 u_2) + u_2^* L_2 B_2 v_2. \end{aligned}$$

So, to demonstrate Theorem 5 in this case is equivalent to prove that $\phi = 0$.

From (68),

$$L_1 v_2 - \sigma_0 u_2 = \lim_{t \rightarrow 0^+} t^{-1} C_1 B_1 v_1(t) \Rightarrow C_1^\dagger (L_1 v_2 - \sigma_0 u_2) = \lim_{t \rightarrow 0^+} t^{-1} C_1^\dagger C_1 B_1 v_1(t).$$

But, since $C_1^\dagger C_1 = I_{n_1}$, we have

$$C_1^\dagger (L_1 v_2 - \sigma_0 u_2) = \lim_{t \rightarrow 0^+} t^{-1} B_1 v_1(t).$$

Thus

$$\phi = u_2^* L_1 B_1^\dagger \lim_{t \rightarrow 0^+} t^{-1} B_1 v_1(t) + u_2^* L_2 v_2 = \lim_{t \rightarrow 0^+} u_2(t)^* L_1 B_1^\dagger \lim_{t \rightarrow 0^+} t^{-1} B_1 v_1(t) + u_2^* L_2 v_2;$$

That is,

$$\phi = \lim_{t \rightarrow 0^+} \frac{u_2(t)^* L_1 B_1^\dagger B_1 v_1(t)}{t} + u_2^* L_2 v_2.$$

because both limits exist. By (34) we find that

$$t u_1^*(t) L_2 P_B + u_2^*(t) L_1 P_B = f(t) v_2(t)^* \Rightarrow t u_1^*(t) L_2 P_B + u_2^*(t) L_1 - u_2^*(t) L_1 B_1^\dagger B_1 = f(t) v_2(t)^*.$$

Therefore

$$u_2^*(t) L_1 B_1^\dagger B_1 = t u_1^*(t) L_2 P_B + u_2^*(t) L_1 - f(t) v_2(t)^*.$$

Consequently

$$\phi = \lim_{t \rightarrow 0^+} \frac{t u_1^*(t) L_2 P_B v_1(t) + u_2^*(t) L_1 v_1(t) - f(t) v_2(t)^* v_1(t)}{t} + u_2^* L_2 v_2,$$

and, as $P_B v_1 = 0$ by (76),

$$\phi = \lim_{t \rightarrow 0^+} \frac{u_2^*(t) L_1 v_1(t) - f(t) v_2(t)^* v_1(t)}{t} + u_2^* L_2 v_2.$$

By (31), $P_C L_1 v_1(t) + t P_C L_2 v_2(t) = f(t) u_1(t)$. Hence we know that $L_1 v_1(t) = f(t) u_1(t) - t P_C L_2 v_2(t) + C_1 C_1^\dagger L_1 v_1(t)$. Since $u_2^* P_C = u_2^*$, it follows that

$$\phi = \lim_{t \rightarrow 0^+} \frac{f(t) u_2(t)^* u_1(t) + u_2(t)^* C_1 C_1^\dagger L_1 v_1(t) - f(t) v_2(t)^* v_1(t)}{t}.$$

But, by (41), we have $u_2(t)^* u_2(t) = v_2(t)^* v_2(t)$. Therefore

$$\phi = \lim_{t \rightarrow 0^+} \frac{u_2(t)^* C_1 C_1^\dagger L_1 v_1(t)}{t} = \lim_{t \rightarrow 0^+} \frac{u_2(t)^* C_1 B_1 B_1^\dagger C_1^\dagger L_1 v_1(t)}{t},$$

because $B_1 B_1^\dagger = I_{n_1}$. Finally, we will apply Lemma 8. Taking $x(t) := u_2(t)$, $y(t) := B_1^\dagger C_1^\dagger L_1 v_1(t)$ and $G = C_1 B_1$, we obtain

$$\lim_{t \rightarrow 0^+} G y(t) = \lim_{t \rightarrow 0^+} C_1 B_1 B_1^\dagger C_1^\dagger L_1 v_1(t) = 0, \quad \lim_{t \rightarrow 0^+} \frac{x(t)^* G}{t} = \lim_{t \rightarrow 0^+} \frac{u_2(t)^* C_1 B_1}{t} = 0,$$

using (70) and that $u_1(t), v_1(t) \rightarrow 0$. Thus, by Lemma 8 we have $\phi = 0$.

If $v_2 = 0$, since by Lema 16(3), $y_1 = 0$, it suffices to repeat the preceding reasoning for the pair (x, y) , with the matrix $D_0 := D - \sigma_0 X Y^\dagger$.

8.2 $u_1 = \alpha u_2$, $v_1 = \alpha v_2$, $x_1 = \beta x_2$, and $y_1 = \beta y_2$, with $\alpha\beta \neq 0$.

Since u_2, v_2 are nonzero vectors, from the proof of Lemma 16 we conclude that $(u_2/\|u_2\|, v_2/\|v_2\|)$ is a pair of singular vectors of $P_C L_1 P_B$ associated with the singular value σ_0 . The same occurs for $(x_2/\|x_2\|, y_2/\|y_2\|)$.

Let (z, w) be a pair of singular vectors of $P_C L_1 P_B$ associated with the simple singular value σ_0 . Hence, as $\|u_2\| = \|v_2\|$ and $\|x_2\| = \|y_2\|$ by (78), we infer that there exist two nonzero complex numbers δ, η such that

$$u = \begin{pmatrix} \delta\alpha z \\ \delta z \end{pmatrix}, v = \begin{pmatrix} \delta\alpha w \\ \delta w \end{pmatrix}, x = \begin{pmatrix} \eta\beta z \\ \eta z \end{pmatrix}, y = \begin{pmatrix} \eta\beta w \\ \eta w \end{pmatrix}.$$

Since v, y are orthogonal, $\bar{\delta}\eta(\bar{\alpha}\beta + 1)w^*w = 0$. Consequently

$$\bar{\alpha}\beta + 1 = 0. \quad (79)$$

On the other hand, applying Lemma 11, for $t \in (0; \varepsilon)$, one has

$$f'(t) = \operatorname{Re}(u^*(t)S_2'(t)v(t)) = \operatorname{Re}\left(\begin{pmatrix} u_1(t)^* & u_2(t)^* \end{pmatrix} \begin{pmatrix} O & P_C L_2 P_B \\ t^{-2}C_1 B_1 & O \end{pmatrix} \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}\right).$$

Since $u_1(t)^* P_C L_2 P_B v_2(t) = u_1(t)^* L_2 v_2(t)$, we get

$$f'(t) = \operatorname{Re}(t^{-2}u_2(t)^* C_1 B_1 v_1(t) + u_1(t)^* L_2 v_2(t)) = t^{-2}u_2(t)^* C_1 B_1 v_1(t) + u_1(t)^* L_2 v_2(t),$$

because (40). As $C_1 B_1 = C_1 B_1 (C_1 B_1)^\dagger C_1 B_1$ and $(C_1 B_1)^\dagger = B_1^\dagger C_1^\dagger$, by Lemma 7-4, we obtain

$$t^{-2}u_2(t)^* C_1 B_1 v_1(t) = \frac{u_2(t)^* C_1 B_1}{t} B_1^\dagger C_1^\dagger \frac{C_1 B_1 v_1(t)}{t}.$$

Thus, from (70) and (68), we see that

$$\lim_{t \rightarrow 0^+} t^{-2}u_2(t)^* C B v_1(t) = (u_1^* L_1 - \sigma_0 v_1^*) B_1^\dagger C_1^\dagger (L_1 v_2 - \sigma_0 u_2).$$

Therefore, as $v_1^* B_1^\dagger = 0$ and $C_1^\dagger u_2 = 0$, we infer that

$$\lim_{t \rightarrow 0^+} f'(t) = u_1^* (L_1 B_1^\dagger C_1^\dagger L_1 + L_2) v_2. \quad (80)$$

Similarly, for $g(t)$ we obtain

$$\lim_{t \rightarrow 0^+} g'(t) = x_1^* (L_1 B_1^\dagger C_1^\dagger L_1 + L_2) y_2. \quad (81)$$

Now, since the functions $f(t), g(t)$ are strictly nonincreasing and f', g' are continuous functions, we see that $f'(t), g'(t)$ are nonpositive. As there exist the limits of $f'(t), g'(t)$ when $t \rightarrow 0^+$, given in (80) and (81), we deduce that

$$u_1^* (L_1 B_1^\dagger C_1^\dagger L_1 + L_2) v_2 \leq 0 \text{ and } x_1^* (L_1 B_1^\dagger C_1^\dagger L_1 + L_2) y_2 \leq 0. \quad (82)$$

Using the expressions obtained to the beginning of this subsection for u, v, x, y , we get

$$\begin{aligned} u_1^* (L_1 B_1^\dagger C_1^\dagger L_1 + L_2) v_2 &= |\delta|^2 \bar{\alpha} z^* (L_1 B_1^\dagger C_1^\dagger L_1 + L_2) w, \\ x_1^* (L_1 B_1^\dagger C_1^\dagger L_1 + L_2) y_2 &= |\eta|^2 \bar{\beta} z^* (L_1 B_1^\dagger C_1^\dagger L_1 + L_2) w. \end{aligned}$$

Thus, from (82) we obtain

$$\bar{\alpha}z^*(L_1B_1^\dagger C_1^\dagger L_1 + L_2)w \leq 0 \text{ and } \bar{\beta}z^*(L_1B_1^\dagger C_1^\dagger L_1 + L_2)w \leq 0.$$

Denote in a short while $\chi := z^*(L_1B_1^\dagger C_1^\dagger L_1 + L_2)w \in \mathbb{C}$. Hence, as $\bar{\alpha}\beta + 1 = 0$ by (79), from the preceding inequalities, we find that

$$-\beta^{-1}\chi \leq 0 \text{ and } \bar{\beta}\chi \leq 0.$$

Consequently, since $\beta \neq 0$, these two inequalities are only possible if $\chi = 0$. That is, we have proved that if (z, w) is a pair of singular vectors of $P_C L_1 P_B$ associated with the singular value σ_0 , then $z^*(L_1B_1^\dagger C_1^\dagger L_1 + L_2)w = 0$. Therefore, for the pair (u, v) one has

$$u_2^*(L_1B_1^\dagger C_1^\dagger L_1 + L_2)v_2 = 0. \quad (83)$$

Next, defining the matrix $D_0 := D - \sigma_0 UV^\dagger$ we are going to prove that 0 is a multiple eigenvalue of $M(\alpha, D_0)$. In a similar way to that of Subsection 8.1, given that

$$\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} \begin{pmatrix} -C_1^\dagger(L_1v_2 - \sigma_0u_2) \\ -\Sigma^{-1}B_2v_2 \\ v_2 \end{pmatrix} = 0,$$

and

$$(-(u_2^*L_1 - \sigma_0v_2^*)B_1^\dagger, -u_2^*C_2\Sigma^{-1}, u_2^*) \begin{pmatrix} 0 & 0 & B_1 \\ 0 & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} = 0,$$

to prove that 0 is a multiple eigenvalue of $M(\alpha, D_0)$, it suffices to see that

$$\phi = (u_2^*L_1 - \sigma_0v_2^*)B_1^\dagger C_1^\dagger(L_1v_2 - \sigma_0u_2) + u_2^*L_2v_2 = 0.$$

That is, as $v_2B_1^\dagger = 0$ and $C_1^\dagger u_2 = 0$, it suffices to see that

$$u_2^*(L_1B_1^\dagger C_1^\dagger L_1 + L_2)v_2 = 0,$$

which is true by (83). *Thus the proof of Theorem 5, which began in Section 5, has just finished.*

9 Equality of the supremum to the limits at 0 and ∞

9.1 When $\lim_{t \rightarrow \infty} \sigma_h(S_2(t))$ is positive

Let us remark that in the part of the proof of Theorem 5 corresponding to the case *When the supremum is the limit at ∞* , Section 6, we have not used the assumption

$$\sup_{t>0} \sigma_h(S_2(t)) = \lim_{t \rightarrow \infty} \sigma_h(S_2(t)).$$

Actually, all we have used is the fact that

$$\lim_{t \rightarrow \infty} \sigma_h(S_2(t)) > 0.$$

Thus, from Proposition 1 and Lemma 14, is easy to prove the following result.

Proposition 17. Let $L := P_C L_2 P_B$. If $h > \text{rank}(L)$ and

$$\sigma_0 := \sigma_{h-\text{rank}(L)} \begin{pmatrix} P_C(I_m - LL^\dagger)L_1 & O \\ O & L_1(I_m - L^\dagger L)P_B \end{pmatrix}$$

is positive, then

$$\min_{\substack{X \in \mathbb{C}^{m \times m} \\ m(0, M(\alpha, X)) \geq 2}} \|X - D\| = \sigma_0.$$

Consequently,

$$\sup_{t>0} \sigma_h(S_2(t)) = \lim_{t \rightarrow \infty} \sigma_h(S_2(t)).$$

Then, following the same reasoning for each $p > \text{rank}(L)$, by Proposition 1 and Lemma 14, we can prove the following proposition.

Proposition 18. (1) Let $L := P_C L_2 P_B$. Assume $p > \text{rank}(P_C L_2 P_B)$. Then

$$\min_{\substack{X \in \mathbb{C}^{m \times m} \\ m(0, M(\alpha, X)) \geq 2}} \|X - D\| \leq \sigma_{p-\text{rank}(L)} \begin{pmatrix} P_C(I_m - LL^\dagger)L_1 & O \\ O & L_1(I_m - L^\dagger L)P_B \end{pmatrix},$$

if this singular value is positive.

(2) For each positive integer q the limit

$$\lim_{t \rightarrow \infty} \sigma_{h+q}(S_2(t)) = \lim_{t \rightarrow \infty} \sigma_{h-\text{rank}(L)+q} \begin{pmatrix} P_C(I_m - LL^\dagger)L_1 & O \\ O & L_1(I_m - L^\dagger L)P_B \end{pmatrix}$$

is equal to σ_0 or to 0.

9.2 When $\lim_{t \rightarrow 0^+} \sigma_h(S_2(t))$ is positive, and $\text{rank}(B_1) < n_1$ or $\text{rank}(C_1) < n_1$

Let us remark that in the part of the proof of Theorem 5 corresponding to the case *When the supremum is the limit at 0^+* , Section 7, we have not used the assumption

$$\sup_{t>0} \sigma_h(S_2(t)) = \lim_{t \rightarrow 0^+} \sigma_h(S_2(t)).$$

Actually, all we have used is the fact that

$$\lim_{t \rightarrow 0^+} \sigma_h(S_2(t)) > 0.$$

Thus, from Proposition 1 and Lemma 13, we can prove the following result.

Proposition 19. Assume that $\text{rank}(B_1) < n_1$ or $\text{rank}(C_1) < n_1$. If $h > \text{rank}(C_1 B_1)$ and

$$\sigma_0 := \sigma_{h-\text{rank}(C_1 B_1)} \begin{pmatrix} P_C L_1 (I_m - (C_1 B_1)^\dagger C_1 B_1) & O \\ O & (I_m - C_1 B_1 (C_1 B_1)^\dagger) L_1 P_B \end{pmatrix}$$

is positive, then

$$\min_{\substack{X \in \mathbb{C}^{m \times m} \\ m(0, M(\alpha, X)) \geq 2}} \|X - D\| = \sigma_0.$$

Consequently,

$$\sup_{t>0} \sigma_h(S_2(t)) = \lim_{t \rightarrow 0^+} \sigma_h(S_2(t)).$$

Then, following the same reasoning for each $p > \text{rank}(L)$, by Proposition 1 and Lemma 13, we can prove the following proposition.

Proposition 20. *Assume that $\text{rank}(B_1) < n_1$ or $\text{rank}(C_1) < n_1$.*

(1) *Suppose that $p > \text{rank}(C_1 B_1)$. Then*

$$\begin{aligned} & \min_{\substack{X \in \mathbb{C}^{m \times m} \\ \mathfrak{m}(0, M(\alpha, X)) \geq 2}} \|X - D\| \\ & \leq \sigma_{p - \text{rank}(C_1 B_1)} \left(\begin{array}{cc} P_C L_1 (I_m - (C_1 B_1)^\dagger C_1 B_1) & O \\ O & (I_m - C_1 B_1 (C_1 B_1)^\dagger) L_1 P_B \end{array} \right), \end{aligned}$$

if this singular value is positive.

(3) *For each positive integer q it follows that the limit*

$$\lim_{t \rightarrow 0^+} \sigma_{h+q}(S_2(t))$$

is equal to σ_0 or to 0.

9.3 When $\lim_{t \rightarrow 0^+} \sigma_h(S_2(t))$ is positive, and $\text{rank}(B_1) = \text{rank}(C_1) = n_1$

Let us assume $\text{rank}(B_1) = \text{rank}(C_1) = n_1$. Following the reasonings in Section 8, it is possible to prove

$$\sup_{t > 0} \sigma_h(S_2(t)) = \lim_{t \rightarrow 0^+} \sigma_h(S_2(t))$$

when this limit is > 0 , *under additional assumptions.*

Thus the following question arises: If the limit

$$\lim_{t \rightarrow 0^+} \sigma_h(S_2(t))$$

is finite and positive, is always true that

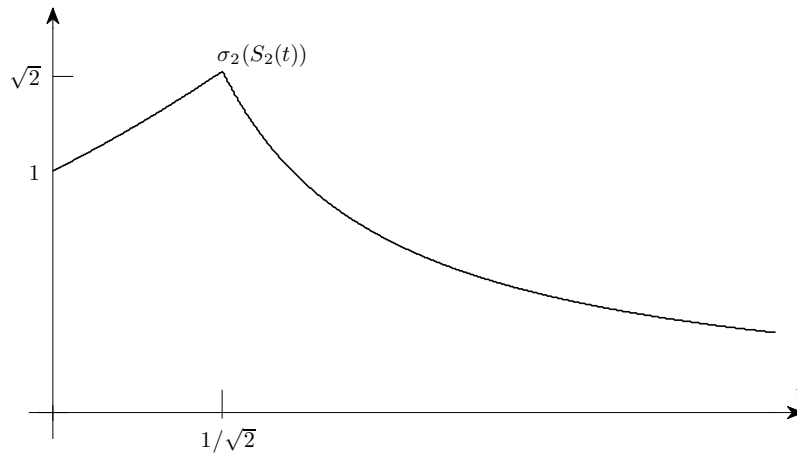
$$\sup_{t > 0} \sigma_h(S_2(t)) = \lim_{t \rightarrow 0^+} \sigma_h(S_2(t))?$$

The answer is negative, as it can be seen in the following counterexample. Consider the matrix of $\mathbb{C}^{3 \times 3}$

$$\left(\begin{array}{cc|cc} 0 & B & 1 & 0 \\ C & D & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array} \right) := \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & t \\ -1/t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right).$$

Then, $h = 2$ and

$$\sigma_2(S_2(t)) = \begin{cases} \sqrt{\frac{t^2 + 2 + t\sqrt{t^2 + 4}}{2}} & \text{if } t \in (0, 1/\sqrt{2}], \\ 1/t & \text{if } t \in [1/\sqrt{2}, \infty). \end{cases}$$



We have

$$\lim_{t \rightarrow 0^+} \sigma_2(S_2(t)) = 1 > 0,$$

but the supremum is attained at $t_0 = 1/\sqrt{2}$ and its value is $\sqrt{2}$.

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