

DERIVATIVES OF THE DIAMETER AND THE AREA OF A CONNECTED COMPONENT OF THE PSEUDOSPECTRA.*

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Abstract. Let $A \in \mathbb{C}^{n \times n}$ and let $\Lambda_\varepsilon(A)$ denote the pseudospectrum of A of level $\varepsilon \geq 0$. Let $\delta(\varepsilon)$ and $a(\varepsilon)$ be the diameter and the area of the connected component of $\Lambda_\varepsilon(A)$ that contains the eigenvalue λ of A of index ν . It is established the relation between the condition number of λ of order $1/\nu$ and the first and second right derivatives of δ and a at $\varepsilon = 0$.

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1. Introduction. Let $A \in \mathbb{C}^{n \times n}$. By $\Lambda(A)$ we denote the spectrum of A . We denote by $\|\cdot\|$ the spectral matrix norm. Let λ be an eigenvalue of A of algebraic multiplicity m . Let us recall that the *spectral variation* of $A' \in \mathbb{C}^{n \times n}$ with respect to the eigenvalue λ of A , denoted by $sv_{(A,\lambda)}(A')$, is defined as the minimum of $\rho \geq 0$ such that the closed disk $\mathcal{D}(\lambda, \rho)$ with center at λ and radius ρ contains at least m eigenvalues (counting multiplicities) of A' . Also, the (*Hölder*) *condition number of the eigenvalue* $\lambda \in \Lambda(A)$ of order $\omega > 0$ is defined by

$$\text{cond}_\omega(A, \lambda) := \lim_{\varepsilon \rightarrow 0^+} \max_{0 < \|X - A\| \leq \varepsilon} \frac{sv_{(A,\lambda)}(X)}{\|X - A\|^\omega}.$$

Note that the number $\text{cond}_\omega(A, \lambda) \in [0, \infty]$ is well defined for each $\omega > 0$; also, there is at most one $\omega > 0$ such that $0 \neq \text{cond}_\omega(A, \lambda) \neq \infty$. See [5, Definition 2.6.3, p. 33]. The *index of an eigenvalue* λ of A is the algebraic multiplicity of λ as a root of the minimal polynomial of A ; let us denote it by $\nu = \nu(\lambda)$. The limit that defines $\text{cond}_{1/\nu}(A, \lambda)$ has sense just for $\omega = 1/\nu$.

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For each $\varepsilon \geq 0$, the ε -pseudospectrum of A is defined by

$$\Lambda_\varepsilon(A) := \bigcup_{\substack{X \in \mathbb{C}^{n \times n} \\ \|X - A\| \leq \varepsilon}} \Lambda(X).$$

For any complex matrix M we denote by $\sigma_1(M) \geq \sigma_2(M) \geq \dots$ its singular values arranged in decreasing order. It is well known that

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : \sigma_n(zI - A) \leq \varepsilon\}.$$

For each $\varepsilon \geq 0$, let $\mathcal{K}_\lambda(\varepsilon)$ be the connected component of the ε -pseudospectrum $\Lambda_\varepsilon(A)$ that contains λ . We denote by $\partial\mathcal{K}_\lambda(\varepsilon)$ the boundary of $\mathcal{K}_\lambda(\varepsilon)$. From [5, Proposition 2.6.5] we deduce that

$$\text{cond}_\omega(A, \lambda) = \lim_{\varepsilon \rightarrow 0^+} \frac{\max_{z \in \partial\mathcal{K}_\lambda(\varepsilon)} |z - \lambda|}{\varepsilon^\omega}.$$

Let us remark that

$$\rho(\varepsilon) := \max_{z \in \partial\mathcal{K}_\lambda(\varepsilon)} |z - \lambda|$$

is the maximum radius of $\mathcal{K}_\lambda(\varepsilon)$ with respect to λ . So,

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\rho(\varepsilon)}{\varepsilon^\omega} = \text{cond}_\omega(A, \lambda).$$

For each $\varepsilon \geq 0$, the diameter of $\mathcal{K}_\lambda(\varepsilon)$ is defined by

$$\delta(\varepsilon) := \sup_{z_1, z_2 \in \mathcal{K}_\lambda(\varepsilon)} |z_1 - z_2|.$$

Next, we consider four examples.

First example. Let $A \in \mathbb{C}^{n \times n}$ be a normal matrix. Then the ε -pseudospectrum of A is the union of the closed disks of radius ε centered at the eigenvalues of A . So, for sufficiently small $\varepsilon \geq 0$, we have

$$\mathcal{K}_\lambda(\varepsilon) = \mathcal{D}(\lambda, \varepsilon).$$

Therefore, $\rho(\varepsilon) = \varepsilon$, and since the eigenvalues of a normal matrix are semisimple (i.e. of index 1),

$$\text{cond}_1(A, \lambda) = \lim_{\varepsilon \rightarrow 0^+} \frac{\varepsilon}{\varepsilon} = 1.$$

The diameter of $\mathcal{D}(\lambda, \varepsilon)$ is 2ε . So, $\delta'(\varepsilon) = 2$ and $\delta'(0^+) = 2$. If we denote by $a(\varepsilon)$ the area of this circle, $a(\varepsilon) = \pi\varepsilon^2$; hence $a'(\varepsilon) = 2\pi\varepsilon, a''(\varepsilon) = 2\pi$. Therefore, $a'(0^+) = 0, a''(0^+) = 2\pi$.

Second example. Let

$$J_2(\lambda, d) = \begin{bmatrix} \lambda & d \\ 0 & \lambda \end{bmatrix}$$

be like a Jordan block, with complex numbers λ, d and $d \neq 0$. In [5, Theorem 5.4.1, p. 74] was proved that for each $\varepsilon \geq 0$,

$$\Lambda_\varepsilon(J_2(\lambda, d))$$

is a closed disk centered at λ and with radius $r_2(\varepsilon) = \max\{r > 0 \mid \sigma_2(J_2(r, |d|)) \leq \varepsilon\}$. As

$$\sigma_2(J_2(r, |d|)) = \sqrt{r^2 + \frac{|d|^2}{2} - \sqrt{r^2|d|^2 + \frac{|d|^4}{4}}},$$

solving the equation in the unknown r

$$r^2 + \frac{|d|^2}{2} - \sqrt{r^2|d|^2 + \frac{|d|^4}{4}} = \varepsilon^2,$$

we find that $r_2(\varepsilon) = \sqrt{\varepsilon^2 + |d|}\varepsilon$. So, $\delta(\varepsilon) = 2\sqrt{\varepsilon^2 + |d|}\varepsilon$. Hence,

$$\delta'(\varepsilon) = \frac{2\varepsilon + |d|}{\sqrt{\varepsilon^2 + |d|}\varepsilon}, \text{ which implies } \delta'(0^+) = \infty.$$

It is obvious that $\nu(\lambda) = 2$. Let us remark that

$$(1.1) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{\delta(\varepsilon)}{\varepsilon^{1/2}} = 2\sqrt{|d|}.$$

Now, let us see that $c := \text{cond}_{1/2}(J_2(\lambda, d), \lambda) = \sqrt{|d|}$. We need some previous considerations. For a general matrix $A \in \mathbb{C}^{n \times n}$, let $\Lambda(A) = \{\lambda_1, \dots, \lambda_r\}$. Let

$$A = \sum_{j=1}^r (\lambda_j P_j + N_j)$$

be the Jordan decomposition of A , where for each $j \in \{1, \dots, r\}$, P_j is the Riesz projector onto the root subspace (or generalized eigenspace) $\mathcal{R}_{\lambda_j}(A)$ of λ_j and along the sum of root subspaces associated with all eigenvalues of A different from λ_j ; and $N_j := (A - \lambda_j I_n)P_j$ is the nilpotent matrix corresponding to λ_j . By [5, Theorem 5.4.4 (viii), p. 78], if $\nu_j := \nu(\lambda_j)$, then

$$(1.2) \quad \text{cond}_{1/\nu_j}(A, \lambda_j) = \|N_j^{\nu_j-1}\|^{1/\nu_j}.$$

In our present example $A = J_2(\lambda, d) = \lambda I_2 + N$, where

$$N = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix},$$

thus, $c = \|N\|^{1/2} = \sqrt{|d|}$. From (1.1), we have

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\delta(\varepsilon)}{\varepsilon^{1/2}} = 2c.$$

The area $a(\varepsilon)$ is given by $\pi(\varepsilon^2 + |d|\varepsilon)$. So, $a'(\varepsilon) = \pi(2\varepsilon + |d|)$; hence, $a'(0^+) = \pi|d| = \pi c^2$.

Third example. This example is a small generalization of the second one. Let A be any n -by- n complex matrix with a unique eigenvalue λ . Moreover, let us assume that $\nu(\lambda) = 2$. If $N := A - \lambda I_n$, then $\Lambda(N) = \{0\}$. So, N is a nilpotent matrix. Thus, $A = \lambda I_n + N$ is the Jordan decomposition of A . Hence, by (1.1), $c := \text{cond}_{1/2}(A, \lambda) = \|N\|^{1/2} = \sqrt{\sigma_1(\lambda I_n - A)}$. In [2, Corollary 20] we have proved that for each $\varepsilon \geq 0$,

$$\Lambda_\varepsilon(A) = \mathcal{D}(\lambda, \sqrt{\varepsilon^2 + \varepsilon \sigma_1(\lambda I_n - A)}).$$

So, $\delta(\varepsilon) = 2\sqrt{\varepsilon^2 + \varepsilon \sigma_1(\lambda I_n - A)}$, $a(\varepsilon) = \pi(\varepsilon^2 + \varepsilon \sigma_1(\lambda I_n - A))$. Therefore,

$$\delta'(\varepsilon) = \frac{2\varepsilon + \sigma_1(\lambda I_n - A)}{\sqrt{\varepsilon^2 + \varepsilon \sigma_1(\lambda I_n - A)}}, \text{ which implies } \delta'(0^+) = \infty,$$

and

$$a'(\varepsilon) = \pi(2\varepsilon + \sigma_1(\lambda I_n - A)), \text{ which implies } a'(0^+) = \pi \sigma_1(\lambda I_n - A) = \pi c^2.$$

Fourth example. Let λ_1, λ_2 be two different complex numbers. Let $A \in \mathbb{C}^{2 \times 2}$ whose eigenvalues are λ_1 and λ_2 . Let us define

$$d(A) := \sqrt{\|A\|_F^2 - |\lambda_1|^2 - |\lambda_2|^2}.$$

The number $d(A)$ is the departure from normality of A with respect to the Frobenius norm $\|A\|_F := \sqrt{\text{tr}(A^*A)}$. In [5, Proposition 5.5.3, p. 80] was proved that for each $\varepsilon > 0$,

$$(1.3) \quad \Lambda_\varepsilon(A) = \mathcal{D}(\lambda_1, \varepsilon) \cup \mathcal{D}(\lambda_2, \varepsilon) \cup \mathcal{M}_{\lambda_1, \lambda_2}(d(A), \varepsilon)$$

where

$$(1.4) \quad \mathcal{M}_{\lambda_1, \lambda_2}(d, \varepsilon) := \{z \in \mathbb{C} : (|z - \lambda_1|^2 - \varepsilon^2)(|z - \lambda_2|^2 - \varepsilon^2) \leq \varepsilon^2 d^2\}, \quad d \geq 0.$$

Let us consider Figure 1.1 that shows the ε -pseudospectrum for the values of $\varepsilon = 1.00, 1.50, 1.75$ of the matrix

$$A_1 := \begin{bmatrix} 1 + 2i & 3 \\ 0 & -1 - 3i \end{bmatrix}.$$

So, $\lambda_1 = 1 + 2i, \lambda_2 = -1 - 3i, \|A_1\|_F = \sqrt{|1 + 2i|^2 + 3^2 + |-1 - 3i|^2} = \sqrt{5 + 9 + 10} = \sqrt{24}$. Thus, $d(A_1) = \sqrt{24 - 5 - 10} = \sqrt{9} = 3$.

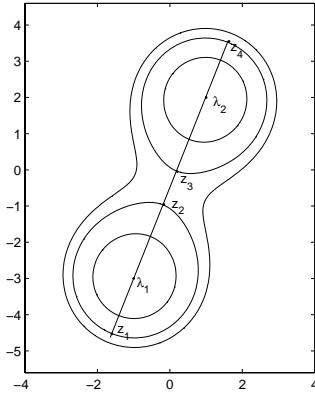


FIG. 1.1. *Pseudospectra of A_1 for $\varepsilon = 1.00, 1.50, 1.75$*

We see that for sufficient small values of ε the ε -pseudospectrum of A_1 has two connected components. Let us fix our attention on $\varepsilon = 1.50$. The figure contains a straight line that passes by λ_1 and λ_2 . This line intersects the boundary $\partial\Lambda_\varepsilon(A_1)$ at the points z_1, z_2, z_3, z_4 . If we consider the eigenvalue λ_2 , the diameter $\delta(\varepsilon)$ of the connected component $\mathcal{K}_{\lambda_2}(\varepsilon)$ is equal to the distance between z_3 and z_4 . From (1.3) and (1.4), we can deduce that

$$\delta(\varepsilon) = \sqrt{1 + \varepsilon^2 + \sqrt{7}\varepsilon} - \sqrt{1 + \varepsilon^2 - \sqrt{7}\varepsilon}.$$

Instead of making a special reasoning for the matrix A_1 , we are going to find a general expression for the diameter $\delta(\varepsilon)$ of $\mathcal{K}_{\lambda_2}(\varepsilon)$ for any matrix $A \in \mathbb{C}^{2 \times 2}$ with eigenvalues λ_1 and λ_2 . By (1.4), the boundary $\partial\Lambda_\varepsilon(A)$ is formed by quasi-Cassini ovals with foci the points λ_1 and λ_2 . In fact, the set $\Lambda_\varepsilon(A)$ is symmetric about the straight line that joints λ_1 and λ_2 . This is a consequence of (1.3) and (1.4). Moreover, $\delta(\varepsilon) = |z_4 - z_3| = |z_2 - z_1|$. Later we will need the condition numbers of λ_1 and λ_2 of

order 1. These number are equal. In fact, by [5, Proposition 5.5.8, p. 83], we have

$$(1.5) \quad \text{cond}_1(A, \lambda_k) = \sqrt{1 + \left(\frac{d(A)}{|\lambda_2 - \lambda_1|} \right)^2}, \quad k = 1, 2.$$

Thus, let $\varepsilon \geq 0$ such that $\Lambda_\varepsilon(A)$ has two connected components. Now, we compute the intersection points z_1, z_2, z_3, z_4 of the line $z(t) := (1-t)\lambda_1 + t\lambda_2, t \in \mathbb{R}$ with the curve $\partial\Lambda_\varepsilon(A)$. By (1.4), this curve is given by the equation

$$(1.6) \quad (|z - \lambda_1|^2 - \varepsilon^2)(|z - \lambda_2|^2 - \varepsilon^2) - \varepsilon^2 d(A)^2 = 0.$$

For determining the values of the parameter t that correspond to the points z_1, z_2, z_3 and z_4 we substitute $z(t)$ into (1.6),

$$\begin{aligned} & |\lambda_1 - \lambda_2|^4 t^4 - 2|\lambda_1 - \lambda_2|^4 t^3 + (|\lambda_1 - \lambda_2|^4 - 2\varepsilon^2|\lambda_1 - \lambda_2|^2) t^2 \\ & + 2\varepsilon^2|\lambda_1 - \lambda_2|^2 t - \varepsilon^2|\lambda_1 - \lambda_2|^2 + \varepsilon^4 - \varepsilon^2 d(A)^2 = 0. \end{aligned}$$

For simplicity, we write $\theta := |\lambda_1 - \lambda_2|$,

$$\theta^4 t^4 - 2\theta^4 t^3 + (\theta^4 - 2\varepsilon^2\theta^2) t^2 + 2\varepsilon^2\theta^2 t - \varepsilon^2\theta^2 + \varepsilon^4 - \varepsilon^2 d(A)^2 = 0.$$

With the command `solve` of MAPLE 13 we find the roots of this equation in t obtaining

$$\begin{aligned} t_1 &= \frac{1}{2} - \frac{1}{2\theta} \sqrt{\theta^2 + 4\varepsilon^2 - 4\varepsilon\sqrt{\theta^2 + d(A)^2}} \\ t_2 &= \frac{1}{2} - \frac{1}{2\theta} \sqrt{\theta^2 + 4\varepsilon^2 + 4\varepsilon\sqrt{\theta^2 + d(A)^2}} \\ t_3 &= \frac{1}{2} + \frac{1}{2\theta} \sqrt{\theta^2 + 4\varepsilon^2 - 4\varepsilon\sqrt{\theta^2 + d(A)^2}} \\ t_4 &= \frac{1}{2} + \frac{1}{2\theta} \sqrt{\theta^2 + 4\varepsilon^2 + 4\varepsilon\sqrt{\theta^2 + d(A)^2}} \end{aligned}$$

where $t_1 < 0 < t_2 < \frac{1}{2} < t_3 < 1 < t_4$. Hence, $z_3 = z(t_3)$ and $z_4 = z(t_4)$. So,

$$\delta(\varepsilon) = |z_4 - z_3| = \frac{1}{2} \sqrt{\theta^2 + 4\varepsilon^2 + 4\varepsilon\sqrt{\theta^2 + d(A)^2}} - \frac{1}{2} \sqrt{\theta^2 + 4\varepsilon^2 - 4\varepsilon\sqrt{\theta^2 + d(A)^2}}$$

Hence,

$$\delta'(\varepsilon) = \frac{2\varepsilon + \sqrt{\theta^2 + d(A)^2}}{\sqrt{\theta^2 + 4\varepsilon^2 + 4\varepsilon\sqrt{\theta^2 + d(A)^2}}} - \frac{2\varepsilon - \sqrt{\theta^2 + d(A)^2}}{\sqrt{\theta^2 + 4\varepsilon^2 - 4\varepsilon\sqrt{\theta^2 + d(A)^2}}}$$

$$\delta'(0) = \frac{2\sqrt{\theta^2 + d(A)^2}}{\sqrt{\theta^2}} = 2\sqrt{1 + \left(\frac{d(A)}{\theta} \right)^2}.$$

By (1.5), we have $\delta'(0) = 2 \operatorname{cond}_1(A, \lambda_2)$.

REMARK 1.1. *We will see that the results in these examples are not casual. In fact, the condition number of λ of order $1/\nu$ is related with the functions δ and a .*

2. Bounds by closed disks. From Theorems 2.6.6 and 5.4.4 of [5] we infer the following theorem.

THEOREM 2.1. *Let λ be an eigenvalue of $A \in \mathbb{C}^{n \times n}$ of index ν . For each $\eta > 0$ there exists an $\varepsilon_\eta > 0$ such that for every $\varepsilon \in (0, \varepsilon_\eta]$,*

$$(2.1) \quad \mathcal{D}(\lambda, ((1 - \eta)\varepsilon)^{1/\nu} c) \subset \mathcal{K}_\lambda(\varepsilon) \subset \mathcal{D}(\lambda, ((1 + \eta)\varepsilon)^{1/\nu} c)$$

c being the condition number of λ of order $1/\nu$.

From (2.1) we deduce that

$$\mathcal{D}(0, ((1 - \eta)\varepsilon)^{1/\nu} c) \subset \mathcal{K}_\lambda(\varepsilon) - \lambda \subset \mathcal{D}(0, ((1 + \eta)\varepsilon)^{1/\nu} c)$$

Considering the homotecy

$$z \mapsto \frac{z}{\varepsilon^{1/\nu}}, \quad z \in \mathbb{C},$$

we see

$$(2.2) \quad \mathcal{D}(0, (1 - \eta)^{1/\nu} c) \subset \frac{\mathcal{K}_\lambda(\varepsilon) - \lambda}{\varepsilon^{1/\nu}} \subset \mathcal{D}(0, (1 + \eta)^{1/\nu} c).$$

3. Bounds by diameters. Since the diameter function is monotone increasing with respect to \subset , by (2.2),

$$(3.1) \quad 2c(1 - \eta)^{1/\nu} \leq \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} \leq 2c(1 + \eta)^{1/\nu}.$$

Thus, we arrive to the following theorem.

THEOREM 3.1. *Let ν be the index of λ , and let c be the condition number of λ of order $1/\nu$. Then*

$$(3.2) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} = 2c.$$

Proof. Let us define the functions

$$(3.3) \quad f_1(\eta) := 1 - (1 - \eta)^{1/\nu},$$

$$(3.4) \quad f_2(\eta) := (1 + \eta)^{1/\nu} - 1,$$

where $\eta \in [0, 1]$. If $\nu = 1$, then

$$\begin{aligned} f_1(\eta) &:= 1 - 1 + \eta = \eta, \\ f_2(\eta) &:= 1 + \eta - 1 = \eta; \end{aligned}$$

so, $f_1(\eta) = f_2(\eta)$ on $[0, 1]$.

If $\nu \geq 2$, we will deduce that $f_2(\eta) \leq f_1(\eta)$. This last inequality is equivalent to

$$(1 + \eta)^{1/\nu} - 1 \leq 1 - (1 - \eta)^{1/\nu}, \quad \eta \in [0, 1],$$

$$(1 + \eta)^{1/\nu} + (1 - \eta)^{1/\nu} \leq 2, \quad \eta \in [0, 1].$$

Let us define

$$g(\eta) := (1 + \eta)^{1/\nu} + (1 - \eta)^{1/\nu}, \quad \eta \in [0, 1].$$

Then

$$g'(\eta) = \frac{1}{\nu} [(1 + \eta)^{1/\nu-1} - (1 - \eta)^{1/\nu-1}].$$

Since $x^\alpha := e^{\alpha \ln x}$, when $\alpha < 0$ the function $x \mapsto x^\alpha$ is decreasing in $(0, \infty)$; therefore, $x \mapsto x^{1/\nu-1}$ is decreasing in $(0, \infty)$. Hence, if $0 < \eta < 1$, we see that $1 - \eta < 1 + \eta$; what implies

$$(1 - \eta)^{1/\nu-1} > (1 + \eta)^{1/\nu-1}.$$

Thus, $g'(\eta) < 0$. So, g is decreasing on $[0, 1]$. Therefore, $g(\eta) < g(0) = 2$. That is, $f_2(\eta) \leq f_1(\eta)$.

Given that

$$\begin{aligned} 2c - 2cf_1(\eta) &= 2c(1 - \eta)^{1/\nu}, \\ 2c + 2cf_2(\eta) &= 2c(1 + \eta)^{1/\nu}, \end{aligned}$$

by (3.1), we have

$$2c - 2cf_1(\eta) \leq \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} \leq 2c + 2cf_2(\eta).$$

For every $\nu \geq 1$, $f_2(\eta) \leq f_1(\eta)$ for $\eta \in [0, 1]$. Then,

$$2c - 2cf_1(\eta) \leq \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} \leq 2c + 2cf_1(\eta),$$

$$-2cf_1(\eta) \leq \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} - 2c \leq 2cf_1(\eta),$$

or

$$\left| \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} - 2c \right| \leq 2cf_1(\eta).$$

But $\lim_{\eta \rightarrow 0^+} f_1(\eta) = 0$ and $f_1(\eta) > 0$ for $\eta > 0$. Thus, fixed an $\eta_0 > 0$, then there exists an $\eta_1 > 0$ such that $2cf_1(\eta_1) < \eta_0$. For this η_1 , there is an $\varepsilon_{\eta_1} > 0$ such that for $\varepsilon \in (0, \varepsilon_{\eta_1}]$,

$$\left| \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} - 2c \right| \leq 2cf_1(\eta) < \eta_0.$$

So, there exists the limit

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}}$$

and it is equal to $2c$. \square

4. Derivatives of the diameter. In this section we relate the right derivative of the diameter δ at 0 with the condition number of the eigenvalue λ , when $\nu = 1$. First, we have the following lemma.

LEMMA 4.1. *Let $A \in \mathbb{C}^{n \times n}$ and λ be an eigenvalue of A of index ν . Let $\delta(\varepsilon)$ be the diameter of the connected component of $\Lambda_\varepsilon(A)$ that contains λ . Let c be the condition number of λ of order $1/\nu$. Then, there exists the limit*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}}$$

and it is equal to

$$\begin{array}{ll} 2c & \text{if } \nu = 1, \\ \infty & \text{if } \nu \geq 2. \end{array}$$

Proof. If $\nu = 1$, Theorem 3.1 implies

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\delta(\varepsilon)}{\varepsilon} = 2c.$$

If $\nu \geq 2$, from the same Theorem,

$$\frac{\delta(\varepsilon)}{\varepsilon} = \frac{\delta(\varepsilon)}{\varepsilon^{1/\nu}} \frac{1}{\varepsilon^{1-1/\nu}} \rightarrow 2c \cdot \infty,$$

when $\varepsilon \rightarrow 0^+$. \square

From this lemma the next theorem follows immediately.

THEOREM 4.2. *Let $A \in \mathbb{C}^{n \times n}$ and λ be an eigenvalue of A of index ν . Let $\delta(\varepsilon)$ be the diameter of the connected component of $\Lambda_\varepsilon(A)$ that contains λ . Let c be the condition number of λ of order $1/\nu$. Then, there exists the right derivative of δ at 0, $\delta'(0^+)$, and*

$$\delta'(0^+) = \begin{cases} 2c & \text{if } \nu = 1, \\ \infty & \text{if } \nu \geq 2. \end{cases}$$

5. Bounds by areas. Since the area function is monotone increasing with respect to \mathbb{C} , by (2.2),

$$(5.1) \quad \pi c^2(1 - \eta)^{2/\nu} \leq \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} \leq \pi c^2(1 + \eta)^{2/\nu},$$

where $a(\varepsilon) := \text{area or Lebesgue measure of } \mathcal{K}_\lambda(\varepsilon)$. Thus, we arrive to the following theorem.

THEOREM 5.1. *Let ν be the index of λ , and let c be the condition number of λ of order $1/\nu$. Then*

$$(5.2) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} = \pi c^2.$$

Proof. Let us define the functions

$$(5.3) \quad \varphi_1(\eta) := 1 - (1 - \eta)^{2/\nu},$$

$$(5.4) \quad \varphi_2(\eta) := (1 + \eta)^{2/\nu} - 1.$$

If $\nu = 1$, then

$$\begin{aligned} \varphi_1(\eta) &= 1 - (1 - \eta)^2 = 2\eta - \eta^2, \\ \varphi_2(\eta) &= (1 + \eta)^2 - 1 = 2\eta + \eta^2; \end{aligned}$$

it is obvious that $\varphi_1(\eta) \leq \varphi_2(\eta)$ when $0 \leq \eta$.

If $\nu = 2$, then

$$\varphi_1(\eta) = 1 - (1 - \eta) = \eta = (1 + \eta) - 1 = \varphi_2(\eta).$$

If $\nu \geq 3$, we will see that

$$\varphi_2(\eta) \leq \varphi_1(\eta), \quad \eta \in [0, 1],$$

or, equivalently,

$$(1 + \eta)^{2/\nu} - 1 \leq 1 - (1 - \eta)^{2/\nu}, \quad \eta \in [0, 1],$$

$$(1 + \eta)^{2/\nu} + (1 - \eta)^{2/\nu} \leq 2, \quad \eta \in [0, 1].$$

Let us define

$$\psi(\eta) := (1 + \eta)^{2/\nu} + (1 - \eta)^{2/\nu}, \quad \eta \in [0, 1].$$

Then

$$\psi'(\eta) = \frac{2}{\nu} \left[(1 + \eta)^{2/\nu-1} - (1 - \eta)^{2/\nu-1} \right].$$

As the function $x \mapsto x^{2/\nu-1}$ is decreasing in $(0, \infty)$, if $0 < \eta < 1$, then $1 - \eta < 1 + \eta$ implies $(1 - \eta)^{2/\nu-1} > (1 + \eta)^{2/\nu-1}$ and $\psi'(\eta) < 0$. In consequence, ψ is decreasing in $[0, 1]$. Thus, for $\eta \in (0, 1]$, $\psi(\eta) < \psi(0) = 2$. Accordingly,

$$\varphi_2(\eta) \leq \varphi_1(\eta), \quad \eta \in [0, 1].$$

By (5.3) and (5.4),

$$\begin{aligned} (1 - \eta)^{2/\nu} &= 1 - \varphi_1(\eta), \\ (1 + \eta)^{2/\nu} &= 1 + \varphi_2(\eta). \end{aligned}$$

Inequalities (5.1) imply

$$(5.5) \quad \pi c^2 - \pi c^2 \varphi_1(\eta) \leq \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} \leq \pi c^2 + \pi c^2 \varphi_2(\eta).$$

The $\nu \leq 2$ case. Since $\varphi_1(\eta) \leq \varphi_2(\eta)$ in $[0, 1]$,

$$-\pi c^2 \varphi_2(\eta) \leq -\pi c^2 \varphi_1(\eta).$$

Hence, by (5.5),

$$(5.6) \quad \pi c^2 - \pi c^2 \varphi_2(\eta) \leq \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} \leq \pi c^2 + \pi c^2 \varphi_2(\eta);$$

which is equivalent to

$$\left| \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} - \pi c^2 \right| \leq \pi c^2 \varphi_2(\eta).$$

But

$$\lim_{\eta \rightarrow 0^+} \varphi_2(\eta) = 0, \quad \varphi_2(\eta) > 0, \quad \forall \eta > 0;$$

so,

$$\lim_{\eta \rightarrow 0^+} \pi c^2 \varphi_2(\eta) = 0.$$

Let us fix an $\eta_0 > 0$, then there exists an $\eta_1 > 0$ such that $\pi c^2 \varphi_2(\eta_1) < \eta_0$. For this η_1 there is an $\varepsilon_{\eta_1} > 0$ such that for $\varepsilon \in (0, \varepsilon_{\eta_1}]$,

$$\left| \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} - \pi c^2 \right| \leq \pi c^2 \varphi_2(\eta_1) < \eta_0.$$

This proves that there exists the limit

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon^{2/\nu}}$$

and it is equal to πc^2 .

The $\nu \geq 3$ case. As $\varphi_2(\eta) \leq \varphi_1(\eta)$ in $[0, 1]$, from (5.5) we deduce that

$$\pi c^2 - \pi c^2 \varphi_1(\eta) \leq \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} \leq \pi c^2 + \pi c^2 \varphi_1(\eta)$$

and, as $\varphi_1(\eta) > 0$ if $\eta > 0$ and

$$\lim_{\eta \rightarrow 0^+} \varphi_1(\eta) = 0,$$

by a reasoning analogous to the former one we infer that there exists the limit

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} = \pi c^2. \quad \square$$

6. Derivatives of the area. In this Section we establish the relation between the first and second right derivatives of a at 0 and the condition number of the eigenvalue λ of order $1/\nu$, when $\nu = 1$ or 2. First, we prove the following lemma.

LEMMA 6.1. *Let $A \in \mathbb{C}^{n \times n}$ and λ be an eigenvalue of A of index ν . Let $a(\varepsilon)$ be the area of the connected component of $\Lambda_\varepsilon(A)$ that contains λ . Let c be the condition number of λ of order $1/\nu$. Then, there exists the limit*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon}$$

and it is equal to

$$\begin{array}{lll} 0 & \text{if} & \nu = 1, \\ \pi c^2 & \text{if} & \nu = 2, \\ \infty & \text{if} & \nu \geq 3. \end{array}$$

Proof. First, let us suppose that $\nu = 1$. Then Theorem 5.1 implies

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon^2} = \pi c^2.$$

Hence,

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0^+} \varepsilon \frac{a(\varepsilon)}{\varepsilon^2} = \left(\lim_{\varepsilon \rightarrow 0^+} \varepsilon \right) \left(\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon^2} \right) = 0 \cdot \pi c^2 = 0.$$

Second, let us assume now that $\nu = 2$. From Theorem 5.1,

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon} = \pi c^2.$$

Finally, when $\nu \geq 3$, there exist $m, M, \varepsilon_0 > 0$ such that

$$m < \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} < M$$

on $(0, \varepsilon_0)$; and as the function

$$\frac{1}{\varepsilon^{(\nu-2)/\nu}}$$

diverges to ∞ when $\varepsilon \rightarrow 0^+$, the product

$$\frac{a(\varepsilon)}{\varepsilon^{2/\nu}} \cdot \frac{1}{\varepsilon^{(\nu-2)/\nu}}$$

diverges to ∞ when $\varepsilon \rightarrow 0^+$. But

$$\frac{a(\varepsilon)}{\varepsilon} = \frac{a(\varepsilon)}{\varepsilon^{2/\nu}} \cdot \frac{1}{\varepsilon^{(\nu-2)/\nu}}.$$

Therefore,

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon} = \infty. \quad \square$$

THEOREM 6.2. *Let $A \in \mathbb{C}^{n \times n}$ and λ be an eigenvalue of A of index ν . Let $a(\varepsilon)$ be the area of the connected component $\Lambda_\varepsilon(A)$ that contains λ . Then, there exists the right derivative of a at 0, $a'(0^+)$, and*

$$a'(0^+) = \begin{cases} 0 & \text{if } \nu = 1, \\ \pi c^2 & \text{if } \nu = 2, \\ \infty & \text{if } \nu \geq 3. \end{cases}$$

Proof. By Lemma 6.1 we deduce that

if $\nu = 1$,

$$a'(0^+) = \lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon) - a(0)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon} = 0;$$

if $\nu = 2$,

$$a'(0^+) = \lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon) - a(0)}{\varepsilon} = \pi c^2;$$

if $\nu \geq 3$,

$$a'(0^+) = \lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon) - a(0)}{\varepsilon} = \infty. \quad \square$$

THEOREM 6.3. *Let $A \in \mathbb{C}^{n \times n}$ and λ be a semisimple eigenvalue. Let $a(\varepsilon)$ be the area of the connected component $\Lambda_\varepsilon(A)$ that contains λ . Let us assume that there exists $a''(0^+)$. Then*

$$a''(0^+) = 2\pi c^2.$$

Proof. Let us define the function

$$A(\varepsilon) := \begin{cases} a(\varepsilon) & \text{if } \varepsilon \geq 0, \\ a(-\varepsilon) & \text{if } \varepsilon < 0. \end{cases}$$

By 6.2, as $\nu = 1$, there exists $A'(0)$ and $A'(0) = 0$. If we suppose that there exists the derivative $A''(0)$, then

$$A''(0) = \lim_{\varepsilon \rightarrow 0} \frac{A(\varepsilon) - 2A(0) + A(-\varepsilon)}{\varepsilon^2}.$$

But, the existence of this limit does not imply the existence of $A''(0)$. See [1, Exercise 5-20.].

When $\varepsilon > 0$, $A(\varepsilon) = a(\varepsilon)$, $A(-\varepsilon) = a(-(-\varepsilon)) = a(\varepsilon)$; therefore, $A(\varepsilon) + A(-\varepsilon) = 2a(\varepsilon)$.

When $\varepsilon < 0$, $A(\varepsilon) = a(-\varepsilon)$, $A(-\varepsilon) = a(-\varepsilon)$; hence, $A(\varepsilon) + A(-\varepsilon) = 2a(-\varepsilon)$.

Then, by Theorem 5.1,

$$\lim_{\varepsilon \rightarrow 0^+} \frac{A(\varepsilon) - 2A(0) + A(-\varepsilon)}{\varepsilon^2} = \lim_{\varepsilon \rightarrow 0^+} \frac{2a(\varepsilon)}{\varepsilon^2} = 2\pi c^2,$$

$$\lim_{\varepsilon \rightarrow 0^-} \frac{A(\varepsilon) - 2A(0) + A(-\varepsilon)}{\varepsilon^2} = \lim_{\varepsilon \rightarrow 0^-} \frac{2a(-\varepsilon)}{(-\varepsilon)^2} = \lim_{\beta \rightarrow 0^+} \frac{2a(\beta)}{\beta^2} = 2\pi c^2.$$

Consequently, there exists the limit

$$\lim_{\varepsilon \rightarrow 0} \frac{A(\varepsilon) - 2A(0) + A(-\varepsilon)}{\varepsilon^2}$$

and is equal to $2\pi c^2$. So, $A''(0) = 2\pi c^2$, and $a''(0^+) = 2\pi c^2$. \square

7. Conjecture. Let λ be a semisimple eigenvalue of a matrix $A \in \mathbb{C}^{n \times n}$. We need the definition of a semialgebraic set $S \subset \mathbb{R}^n$ and of a semialgebraic function $f : S \rightarrow \mathbb{R}$. These concepts can be seen in [5, Chapter 3, p. 39]. A classical reference is [3, Chapter 2, p. 23].

CONJECTURE 7.1. *The functions $\delta, a: [0, \infty) \rightarrow \mathbb{R}$ are semialgebraic.*

The following theorem is proved in [4, Lemma 3.1 (ii)]

THEOREM 7.2 (de l'Hôpital inverse rule). *If $f, g: [0, \infty) \rightarrow \mathbb{R}$ are semialgebraic functions, $f(0) = g(0) = 0$, and there is an $\varepsilon_0 > 0$ such that $g'(\varepsilon) > 0$ for $\varepsilon \in (0, \varepsilon_0)$, then*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{f(\varepsilon)}{g(\varepsilon)} = \ell \in \mathbb{R} \implies \lim_{\varepsilon \rightarrow 0^+} \frac{f'(\varepsilon)}{g'(\varepsilon)} = \ell.$$

If Conjecture 7.1 were true, then

$$\pi c^2 = \lim_{\varepsilon \rightarrow 0^+} \frac{a(\varepsilon)}{\varepsilon^2} = \lim_{\varepsilon \rightarrow 0^+} \frac{a'(\varepsilon)}{2\varepsilon} = \lim_{\varepsilon \rightarrow 0^+} \frac{a''(\varepsilon)}{2}.$$

Thus, the derivative $a''(0^+)$ would exist and it would be equal to $2\pi c^2$. because $a''(0^+) = \lim_{\varepsilon \rightarrow 0^+} a''(\varepsilon)$.

As the area of a region can be expressed by means of a line integral, and taking into account that a parametric integral is derivable with respect to the parameter when the integrand is, the derivatives $a'(\varepsilon)$ and $a''(\varepsilon)$ exist for sufficiently small $\varepsilon > 0$.

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