Identical pseudospectra of any geometric multiplicity[*]

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Abstract

Let $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{m\times m}$ where $n \geq m$ and B has t nontrivial invariant factors. Let us assume that A and B have the same set of ε-pseudoeigenvalues of geometric multiplicity $\geq k$ for each $\varepsilon > 0$ and each k. We prove that the last t invariant factors of A are equal to the nontrivial invariant factors of B. If $n = m$, the matrices A and B are similar.

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1 Introduction

Let $M \in \mathbb{C}^{N\times N}$. Let $\Lambda(M)$ denote the spectrum of M and let $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_N(M)$ denote the singular values of M arranged in decreasing order. We denote by $O_{p\times q}$ the $p \times q$ zero matrix when $p, q \geq 2$, but we write 0 for row or column zero vectors. We write $\mathrm{GL}_N(\mathbb{C})$ for the group of invertible matrices of $\mathbb{C}^{N\times N}$. For any real number $\varepsilon \geq 0$, let $\Lambda_\varepsilon(A)$ denote the set

$$\bigcup_{\substack{X\in\mathbb{C}^{N\times N}\\ \|X-M\|\leq\varepsilon}} \Lambda(X),$$

where $\|\cdot\|$ is the spectral norm. This set is called the *ε-pseudospectrum* of M, and its elements are called the *ε-pseudoeigenvalues* of M.

For $z \in \mathbb{C}$ we denote by $\mathrm{gm}(z, M)$ the geometric multiplicity of z as eigenvalue of M. If $z \notin \Lambda(M)$, we agree $\mathrm{gm}(z, M) := 0$. Let k be integer, $1 \leq k \leq n$,

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let $\Lambda_k^{(g)}(M)$ denote the set of $z \in \Lambda(M)$ such that $\mathrm{gm}(z, M) \geq k$. For $\varepsilon \geq 0$, the *ε-pseudospectrum of M of geometric multiplicity* $\geq k$ is defined by

$$\Lambda_{\varepsilon,k}^{(g)}(M) := \bigcup_{\substack{X \in \mathbb{C}^{N\times N} \\ \|X-M\| \leq \varepsilon}} \Lambda_k^{(g)}(X).$$

The main result in this paper is the following one.

Theorem 1 (Main result)**.** *Let $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{m\times m}$. Suppose that $n \geq m$ and let*

$$g_i(\lambda)|g_{i+1}(\lambda)|\cdots|g_{m-1}(\lambda)|g_m(\lambda)$$

be the nontrivial invariant factors of B. Assume that for each $\varepsilon > 0$ and $k = 1, 2, \ldots, m-i+1$,

$$\Lambda_{\varepsilon,k}^{(g)}(A) = \Lambda_{\varepsilon,k}^{(g)}(B). \tag{1}$$

Then the last $m-i+1$ invariant factors of A,

$$f_{n-m+i}(\lambda)|f_{n-m+i+1}(\lambda)|\cdots|f_{n-1}(\lambda)|f_n(\lambda),$$

are nontrivial, and

$$f_n(\lambda) = g_m(\lambda),\ f_{n-1}(\lambda) = g_{m-1}(\lambda),\ \ldots,\ f_{n-m+i}(\lambda) = g_i(\lambda).$$

As a consequence, we have the next result, which gives us sufficient conditions for the ordinary similarity of matrices in terms of the ε-pseudospectrum of geometric multiplicity $\Lambda_{\varepsilon,k}^{(g)}(M)$.

Corollary 2 (Sufficient condition for similarity)**.** *Let $A, B \in \mathbb{C}^{n\times n}$. Assume that A has s nontrivial invariant factors (counting repetitions). Suppose that for each $\varepsilon > 0$ and $k = 1, 2, \ldots, s$,*

$$\Lambda_{\varepsilon,k}^{(g)}(A) = \Lambda_{\varepsilon,k}^{(g)}(B).$$

Then A and B are similar.

Remark 1. Let us notice that if A and B are similar and both matrices are normal, we have $\Lambda_{\varepsilon,k}^{(g)}(A) = \Lambda_{\varepsilon,k}^{(g)}(B)$. This is not true in general.

Theorem 1 was inspired by the Fact 5(b), page 16-2 in Chapter 16 on pseudospectra by M. Embree in [4]. This Fact says that if A and B have the same *ordinary* ε-pseudospectrum for every $\varepsilon > 0$, then A and B have the same minimal polynomial. But the minimal polynomial is the last invariant factor. The *ordinary* ε-pseudospectrum of a matrix M is our $\Lambda_{\varepsilon,1}^{(g)}(M) = \Lambda_\varepsilon(M)$.

M. F. Bourque and T. Ransford in [1], showed that: (a) $A, B \in \mathbb{C}^{2\times 2}$ satisfy (1) if and only if A is unitarily similar to B; (b) $A, B \in \mathbb{C}^{3\times 3}$ satisfy (1) if and only if A is unitarily similar to B or to its transpose; (c) there exist $A, B \in \mathbb{C}^{4\times 4}$ that satisfy (1) such that $\|A^2\| \neq \|B^2\|$, this implies that A is not unitarily similar either to B or to its transpose.

The organization of this paper is a follows: in Section 2 we will introduce the results used in the article. Given $M \in \mathbb{C}^{N\times N}$ and z_0 an eigenvalue of M, we will analyze the asymptotic behavior of the singular values of the resolvent matrix $(zI_N - M)^{-1}$ when $z \to z_0$ in Section 3. We will prove Theorem 1 in Section 4. Finally, by trying to analyze to what extent pseudospectra $\Lambda^{(g)}_{\varepsilon,k}(M)$ are determined by the invariant factors of M, we have managed to describe these sets for the matrices M with quadratic minimal polynomial by using the reference [2], in Section 5.

2 Preliminary results

In this section, we will introduce some preliminary results that will be used in this paper. We will begin with some properties of the pseudospectra of multiplicity $\geq k$. For the first, we need the following result which is a consequence of the singular value decomposition.

Proposition 3. *Let $M \in \mathbb{C}^{N\times N}$, let $z \in \mathbb{C}$ and k be an integer $1 \leq k \leq N$. Then*

$$\min_{\mathrm{gm}(z,Y)\geq k} \|Y - M\| = \sigma_{N-k+1}(zI_N - M).$$

Proposition 4. *Let $M \in \mathbb{C}^{N\times N}$. For $\varepsilon \geq 0$,*

$$\Lambda^{(g)}_{\varepsilon,k}(M) = \{z \in \mathbb{C} : \sigma_{N-k+1}(zI_N - M) \leq \varepsilon\}.$$

Proof. Let $z \in \Lambda^{(g)}_{\varepsilon,k}(M)$. Then, there exists a matrix X such that $\|X-M\| \leq \varepsilon$ and $z \in \Lambda^{(g)}_{k}(X)$. Now, by Proposition 3, we have

$$\sigma_{N-k+1}(zI_N - M) \leq \|X - M\| \leq \varepsilon.$$

Reciprocally, if $\sigma_{N-k+1}(zI_N - M) \leq \varepsilon$ then

$$\min_{\mathrm{gm}(z,Y)\geq k} \|Y - M\| \leq \varepsilon.$$

Therefore there exists a matrix X such that $\|X - M\| \leq \varepsilon$ and $z \in \Lambda^{(g)}_{k}(X)$.

□

Proposition 5. *Let $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{m\times m}$. Then the following two assertions are equivalent.*

(a) *For $\varepsilon > 0$, $\Lambda^{(g)}_{\varepsilon,k}(A) = \Lambda^{(g)}_{\varepsilon,k}(B)$.*

(b) *For each $z \in \mathbb{C}$, $\quad \sigma_{n-k+1}(zI_n - A) = \sigma_{m-k+1}(zI_m - B)$.*

Proof. The implication (b)⇒(a) is straightforward by Proposition 4. Now let us suppose (b) is not true. That is to say, there exist a complex number z_0 and a real number ε_1 such that

$$\sigma_{n-k+1}(z_0I_n - A) > \varepsilon_1 > \sigma_{m-k+1}(z_0I_m - B),$$

for example. Then, by Proposition 4, $z_0 \in \Lambda^{(g)}_{\varepsilon_1,k}(B)$ but $z_0 \notin \Lambda^{(g)}_{\varepsilon_1,k}(A)$. This contradicts (a).

□

The next property is immediately deduced from the Proposition 4 and [4, Fact 3(a), page 16-2].

Proposition 6. *Let $M \in \mathbb{C}^{N\times N}$. For $\varepsilon \geq 0$,*

(1) *if $U \in \mathbb{C}^{N\times N}$ is an unitary matrix, then $\Lambda_{\varepsilon,k}^{(\mathrm{g})}(M) = \Lambda_{\varepsilon,k}^{(\mathrm{g})}(U^*MU)$.*

(2) *for $\alpha \in \mathbb{C}$, $\Lambda_{\varepsilon,k}^{(\mathrm{g})}(\alpha I_N + M) = \alpha + \Lambda_{\varepsilon,k}^{(\mathrm{g})}(M)$.*

Let A, B be two n-square complex matrices. The matrices A and B are said to be *unitarily similar* if there is a unitary matrix U of order n such that $B = U^*AU$.

The next result, which can be seen in [2, Theorem 3], gives a canonical form for the unitary similarity, for matrices with quadratic minimal polynomial.

Proposition 7. *Two matrices $A, B \in \mathbb{C}^{n\times n}$ with quadratic minimal polynomial, are unitary similar if and only if they have the same eigenvalues and the same singular values.*

Let us represent by $\nu(A) = \dim \mathrm{Ker}(A)$. As an immediate consequence of the previous Proposition, we deduce the next result.

Proposition 8. *Let $M \in \mathbb{C}^{N\times N}$ a matrix whose singular values are $\sigma_1 \geq \cdots \geq \sigma_q > \sigma_{q+1} = \cdots = \sigma_N = 0$. Let us suppose that q, r are nonnegative integers such that $N = 2q + r$.*
(a) *Let us assume that the minimal polynomial of M is λ^2. Let us suppose $\nu(M) = q + r$. Then M is unitarily similar to a matrix of the form*

$$\begin{pmatrix} 0 & \sigma_1 \\ 0 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \sigma_q \\ 0 & 0 \end{pmatrix} \oplus O_{r\times r}. \tag{2}$$

(b) *Let us assume that the minimal polynomial of M is $\lambda(\lambda - \alpha)$, with $\alpha \neq 0$, $\nu(\alpha I_N - M) = q$, $\nu(M) = q + r$, then M is unitarily similar to a matrix of the form*

$$\begin{pmatrix} 0 & s_1 \\ 0 & \alpha \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & s_q \\ 0 & \alpha \end{pmatrix} \oplus O_{r\times r} \tag{3}$$

where $s_i = \sqrt{\sigma_i^2 - \mid \alpha \mid^2}$.

Remark 2. Now let us notice that, by the Proposition 6 for determining the set $\Lambda_{\varepsilon,k}^{(\mathrm{g})}(M)$ it is enough to consider M in the form (2) or (3), if the minimal polynomial of M is quadratic.

The next propositions will be used in Section 5.

Proposition 9. *Let $a \geq 0$. Then the function*

$$f(x) = x^2 - x\sqrt{x^2 + a}$$

is decreasing on $[0, \infty)$.

Proof. As

$$f'(x) = \frac{2x\sqrt{x^2+a} - 2x^2 - a}{\sqrt{x^2+a}},$$

then $f'(x) \leq 0$ if and only if $2x\sqrt{x^2+a} \leq 2x^2 + a$. Then, as we have $a \geq 0$, by squaring, we obtain

$$f'(x) \leq 0 \Leftrightarrow 4x^2(x^2+a) \leq (2x^2+a)^2 \Leftrightarrow 0 \leq a^2.$$

□

Proposition 10. *Let $a > 0$. Then the function*

$$f(x) = 2x - a\sqrt{4x + a^2}$$

is strictly increasing on $[0, \infty)$.

Proof. As

$$f'(x) = 2 - \frac{2a}{\sqrt{4x+a^2}},$$

then

$$f'(x) > 0 \Leftrightarrow 1 > \frac{a}{\sqrt{4x+a^2}},$$

which is true for each $x > 0$.

□

Proposition 11. *Let $a^2 \geq b \geq 0$. Then the function*

$$f(x) = x^2 - \sqrt{(x^2+a)^2 - b}$$

is decreasing on $[0, \infty)$.

Proof. For $x > 0$,

$$f'(x) = 2x\,\frac{\sqrt{(x^2+a)^2-b} - x^2 - a}{\sqrt{(x^2+a)^2-b}};$$

since $\sqrt{(x^2+a)^2 - b} > 0$, it is deduced that

$$f'(x) \leq 0 \Leftrightarrow \sqrt{(x^2+a)^2-b} \leq x^2 + a,$$

which is true.

□

3 Orders of infinity of the singular values of a resolvent matrix

Let a matrix $M \in \mathbb{C}^{N\times N}$ and z_0 an eigenvalue of M, in this section we will study the asymptotic behavior of the singular values of the resolvent matrix $(zI_N - M)^{-1}$, when $z \to z_0$. To that end, we need the next notations.

Let us denote for $\mathbb{R}_+ := \{x \in \mathbb{R}\colon x > 0\}$. Given $z \in \mathbb{C}$ and $V'(z_0) \subset \mathbb{C}$ a deleted neighborhood of z_0, we consider the set

$$\mathcal{F} := \{f : V'(z_0) \to \mathbb{R}_+\}.$$

Then, we have the following definition.

Definition 1. Let $f, g \in \mathcal{F}$. If there are constants $\delta, \Delta, d > 0$ such that for every $z \in B'(z_0, d)$ (open deleted ball centered at z_0 and radius d)

$$\delta \leq \frac{f(z)}{g(z)} \leq \Delta,$$

we write (with Hardy's notation [3])

$$f(z) \asymp g(z) \qquad (\text{when } z \to z_0).$$

The relation $\asymp$ is an equivalence relation. Moreover, it is immediate to prove the next properties.

Proposition 12. *With the previous notations, we have*

(1) *Let $f \in \mathcal{F}$ such that there exists* $\lim_{z \to z_0} f(z) \neq 0$ *and is finite. Then*

$$f(z) \asymp 1 \text{ or } f(z) \asymp \frac{1}{\mid z - z_0 \mid^0} \qquad (z \to z_0).$$

(2) *If j, k are integers ≥ 0 and*

$$\frac{1}{\mid z - z_0 \mid^j} \asymp \frac{1}{\mid z - z_0 \mid^k} \qquad (z \to z_0),$$

then $j = k$.

(3) *Let $f, g \in \mathcal{F}$. If there are constants $c_1, c_2, d > 0$ such that for $z \in B'(z_0, d)$*

$$c_1 g(z) \leq f(z) \leq c_2 g(z),$$

then

$$g(z) \asymp f(z) \qquad (z \to z_0).$$

Let z_0 be an eigenvalue of a matrix $M \in \mathbb{C}^{N \times N}$, and let

$$(\lambda - z_0)^{n_1}, \ldots, (\lambda - z_0)^{n_t}, \text{ where } n_1 \geq \cdots \geq n_t > 0,$$

be the elementary divisors of M associated with z_0. The decreasing sequence $(n_1, \ldots, n_t)$ is said to be the *Segre's partition* of z_0 w.r.t. the matrix M, and it is denoted by $\mathrm{s}(z_0, M)$. The integer t is called the *length* of this partition. If it is convenient we add to $(n_1, \ldots, n_t)$ a tail of zeros $(n_1, \ldots, n_t, n_{t+1}, \ldots, n_N)$, i.e. with $n_{t+1} = \cdots = n_N = 0$.

With the previous notations, the main result of this section is the next.

Theorem 13. *Let $M \in \mathbb{C}^{N \times N}$ and $z_0 \in \Lambda(M)$ with Segre's partition $\mathrm{s}(z_0, M) = (n_1, n_2 \ldots, n_N)$ where $n_1 \geq n_2 \cdots \geq n_N \geq 0$. Then for $j = 1, \ldots, N$,*

$$\sigma_j \left[(zI_N - M)^{-1} \right] \asymp \frac{1}{\mid z - z_0 \mid^{n_j}} \qquad (z \to z_0).$$

For the prove of Theorem 13, we need some previous results. The first one can be seen in [6].

Lemma 14. *Let $M_1, M_2, M_3 \in \mathbb{C}^{N\times N}$. Then, for $k = 1, 2, \ldots, N$*

$$\sigma_N(M_1)\sigma_k(M_2)\sigma_N(M_3) \leq \sigma_k(M_1M_2M_3) \leq \|M_1\|\|M_3\|\sigma_k(M_2).$$

With this result, we prove the next one.

Lemma 15. *Let $M \in \mathbb{C}^{N\times N}$, $P \in \mathrm{GL}_N(\mathbb{C})$ and $z_0 \in \mathbb{C}$. Then, for $j = 1, 2, \ldots, N$*

$$\sigma_j\left[(zI_N - M)^{-1}\right] \asymp \sigma_j\left[(zI_N - P^{-1}MP)^{-1}\right] \qquad (z \to z_0).$$

Proof. Let $z \in \mathbb{C}\backslash\Lambda(M)$. Then, since $(zI_N - P^{-1}MP)^{-1} = P^{-1}(zI_N - M)^{-1}P$, by Lemma 14 we have

$$\sigma_N(P)\sigma_N(P^{-1})\sigma_j\left[(zI_N - M)^{-1}\right] \leq \sigma_j\left[(zI_N - P^{-1}MP)^{-1}\right] \leq$$
$$\leq \|P\|\|P^{-1}\|\sigma_j\left[(zI_N - M)^{-1}\right],$$

for $j = 1, 2, \ldots, N$. Therefore, by Proposition 12(3), we deduce that

$$\sigma_j\left[(zI_N - M)^{-1}\right] \asymp \sigma_j\left[(zI_N - P^{-1}MP)^{-1}\right] \qquad (z \to z_0).$$

□

Lemma 16 (Jordan block)**.** *Given an integer $k > 0$, for each $z \in \mathbb{C}$ let us consider the $k \times k$ block*

$$zI_k - J_k(0) = \begin{pmatrix} z & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -1 \\ 0 & \cdots & \cdots & z \end{pmatrix}.$$

Then

(1)

$$\sigma_1\left[(zI_k - J_k(0))^{-1}\right] \asymp \frac{1}{|z|^k} \qquad (z \to 0).$$

(2) *For $j = 2, \ldots, k$,*

$$\sigma_j\left[(zI_k - J_k(0))^{-1}\right] \asymp 1 \qquad (z \to 0).$$

Proof. (1) As for $z \neq 0$

$$(zI_k - J_k(0))^{-1} = \begin{pmatrix} \frac{1}{z} & \frac{1}{z^2} & \cdots & \frac{1}{z^k} \\ 0 & \frac{1}{z} & \cdots & \frac{1}{z^{k-1}} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{z} \end{pmatrix},$$

then

$$\sigma_1\left[(zI_k - J_k(0))^{-1}\right] = \|(zI_k - J_k(0))^{-1}\| \geq \frac{1}{|z|^k}. \tag{4}$$

On the other hand, if $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm, since $\|\cdot\| \leq \|\cdot\|_{\mathrm{F}}$, then

$$\|(zI_k - J_k(0))^{-1}\|^2 \leq \|(zI_k - J_k(0))^{-1}\|_{\mathrm{F}}^2 = \frac{k}{|z|^2} + \cdots + \frac{2}{|z|^{2k-2}} + \frac{1}{|z|^{2k}}.$$

Now, if $0 < |z| < 1$, then $|z|^{2i} \geq |z|^{2k}$, for $i = 1, 2, \ldots, k$. Therefore,

$$\|(zI_k - J_k(0))^{-1}\|^2 \leq \frac{1}{|z|^{2k}}(k + \cdots + 2 + 1) = \frac{k(k+1)}{2|z|^{2k}}$$

With this inequality and the inequality (4), we deduce that

$$\frac{1}{|z|^k} \leq \sigma_1\left[(zI_k - J_k(0))^{-1}\right] \leq \frac{\sqrt{k(k+1)/2}}{|z|^k},$$

when $0 < |z| < 1$. These two last inequalities together with the Proposition 12(3) prove (1).

(2) Now let us observe that, when $z \to 0$, it results that $zI_k - J_k(0) \to -J_k(0)$. Therefore, since

$$J_k(0)^{\mathrm{T}} J_k(0) = \begin{pmatrix} 0 & 0 \\ 0 & I_{k-1} \end{pmatrix},$$

the singular values of $J_k(0)$ are

$$\sigma_1(J_k(0)) = \cdots = \sigma_{k-1}(J_k(0)) = 1, \quad \sigma_k(J_k(0)) = 0.$$

Therefore, if $z \neq 0$ and $j = 2, \ldots, k$ we obtain that

$$\sigma_j\left[(zI_k - J_k(0))^{-1}\right] = \frac{1}{\sigma_{k-j+1}(zI_k - J_k(0))},$$

then

$$\lim_{z\to 0} \sigma_j\left[(zI_k - J_k(0))^{-1}\right] = \frac{1}{\sigma_{k-j+1}(-J_k(0))} = 1.$$

For that reason and Proposition 12(1), $\sigma_j\left[(zI_k - J_k(0))^{-1}\right] \asymp 1$ when $z \to 0$.

□

Lemma 17. *Let $L_1 \in \mathbb{C}^{p\times p}$ and z_0 be a complex number such that $z_0 \notin \Lambda(L_1)$. Then, for $j = 1, 2, \ldots, p$,*

$$\sigma_j\left[(zI_p - L_1)^{-1}\right] \asymp 1 \qquad (z \to z_0).$$

Proof. As long as z is sufficiently close to z_0 the matrix $zI_p - L_1$ is invertible. Then, for $j = 1, 2, \ldots, p$, the limit

$$\lim_{z\to z_0} \sigma_j\left[(zI_p - L_1)^{-1}\right] = \lim_{z\to z_0} \frac{1}{\sigma_{p-j+1}(zI_p - L_1)} = \frac{1}{\sigma_{p-j+1}(z_0 I_p - L_1)}$$

is nonzero and finite. Therefore the Lemma is deduced from Proposition 12(1).

□

We are ready to show Theorem 13.

Proof Theorem 13. Let us suppose that

$$n_1 \geq n_2 \geq \cdots \geq n_t > n_{t+1} = \cdots = n_N = 0.$$

Let us denote $N_1 := \sum_{k=1}^{t} n_k$ and $N_2 := N - N_1$. Now, let $P \in \mathrm{GL}_N(\mathbb{C})$ be such that

$$L := P^{-1}MP = \begin{pmatrix} \bigoplus_{k=1}^{t} J_{n_k}(z_0) & O_{N_1\times N_2} \\ O_{N_2\times N_1} & L_1 \end{pmatrix},$$

where $J_{n_k}(z_0)$ is a $n_k \times n_k$ Jordan block associated with z_0, $L_1 \in \mathbb{C}^{N_2\times N_2}$ and $z_0 \notin \Lambda(L_1)$. Then, by Lemma 15, for $j = 1, 2, \ldots, N$

$$\sigma_j\left[(zI_N - L)^{-1}\right] \asymp \sigma_j\left[(zI_N - M)^{-1}\right] \qquad (z \to z_0).$$

On the other hand, since

$$(zI_N - L)^{-1} = \begin{pmatrix} \bigoplus_{k=1}^{t} (zI_{n_k} - J_{n_k}(z_0))^{-1} & O_{N_1\times N_2} \\ O_{N_1\times N_1} & (zI_{N_2} - L_1)^{-1} \end{pmatrix},$$

then, from Lemma 16, we have

$$\sigma_j\left[\bigoplus_{k=1}^{t} (zI_{n_k} - J_{n_k}(z_0))^{-1}\right] \asymp \frac{1}{\mid z - z_0 \mid^{n_j}} \quad (z \to z_0), \qquad j = 1, 2 \ldots, t,$$

$$\sigma_j\left[\bigoplus_{k=1}^{t} (zI_{n_k} - J_{n_k}(z_0))^{-1}\right] \asymp 1 \qquad (z \to z_0), \quad j = t+1, \ldots, N_1,$$

and, from Lemma 17, for $j = 1, 2 \ldots, N_2$,

$$\sigma_j\left[(zI_{N_2} - L_1)^{-1}\right] \asymp 1 \qquad (z \to z_0).$$

Then, when $z \to z_0$, we deduce that

$$\sigma_j\left[(zI_N - L)^{-1}\right] \asymp \begin{cases} \frac{1}{|z-z_0|^{n_j}}, & j = 1, 2 \ldots, t, \\ 1, & j = t+1, \ldots, N. \end{cases}$$

This completes the proof.

□

4 Proof of the main result

In this section, we will prove the main result of this paper.

Proof of Theorem 1. In the first place, since $\Lambda^{(\mathrm{g})}_{\varepsilon,1}(A) = \Lambda_\varepsilon(A)$, and $\lim_{\varepsilon\to 0^+} \Lambda_\varepsilon(A) = \Lambda(A)$ with respect to the Hausdorff metric [5, Corollary 2.3.8], then

$$\Lambda(A) = \Lambda(B) = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}.$$

Secondly, for each $\varepsilon > 0$ and $k = 1, 2, \ldots, m-i+1$, we have $\Lambda^{(g)}_{\varepsilon,k}(A) = \Lambda^{(g)}_{\varepsilon,k}(B)$. Then by Proposition 5, we deduce that $\sigma_{n-k+1}(zI_n - A) = \sigma_{m-k+1}(zI_m - B)$ for every complex z. Therefore, when $z \notin \Lambda(A)$, we have

$$\sigma_j\left[(zI_n - A)^{-1}\right] = \sigma_j\left[(zI_m - B)^{-1}\right] \quad j = 1, 2, \ldots, m-i+1. \tag{5}$$

Let

$$f_1(\lambda) \mid \cdots \mid f_{n-1}(\lambda) \mid f_n(\lambda),$$
$$g_1(\lambda) \mid \cdots \mid g_{m-1}(\lambda) \mid g_m(\lambda)$$

be the chains of invariant factors of A and B, respectively. For $h = 1, 2 \ldots, p$, let

$$\mathrm{s}(\lambda_h, A) = (\mathrm{s}_1(\lambda_h, A), \mathrm{s}_2(\lambda_h, A), \ldots, \mathrm{s}_n(\lambda_h, A)),$$
$$\mathrm{s}(\lambda_h, B) = (\mathrm{s}_1(\lambda_h, B), \mathrm{s}_2(\lambda_h, B), \ldots, \mathrm{s}_m(\lambda_h, B))$$

be the Segre's partition of λ_h w.r.t. A and B, respectively. Then, by Theorem 13, we have

$$\begin{aligned} \sigma_j\left[(zI_n - A)^{-1}\right] &\asymp \tfrac{1}{|z-\lambda_h|^{\mathrm{s}_j(\lambda_h, A)}}, \\ \sigma_j\left[(zI_n - B)^{-1}\right] &\asymp \tfrac{1}{|z-\lambda_h|^{\mathrm{s}_j(\lambda_h, B)}}, \end{aligned} \qquad (z \to \lambda_h).$$

Then, by equation (5), we deduce that

$$\frac{1}{\mid z - \lambda_h \mid^{\mathrm{s}_j(\lambda_h, A)}} \asymp \sigma_j\left[(zI_n - A)^{-1}\right] = \sigma_j\left[(zI_n - A)^{-1}\right] \asymp \frac{1}{\mid z - \lambda_h \mid^{\mathrm{s}_j(\lambda_h, B)}},$$

for $j = 1, 2, \ldots, m-i+1$ and whenever $z \to \lambda_h$. Therefore, Proposition 12(2),

$$\mathrm{s}_j(\lambda_h, A) = \mathrm{s}_j(\lambda_h, B), \quad j = 1, 2, \ldots, m-i+1.$$

Then

$$f_n(\lambda) = g_m(\lambda), f_{n-1}(\lambda) = g_{m-1}(\lambda), ..., f_{n-m+i}(\lambda) = g_i(\lambda).$$

□

5 Matrix with quadratic minimal polynomial

In this section, we will analyze the sets $\Lambda^{(g)}_{\varepsilon,k}(M)$, by assuming that M has a quadratic minimal polynomial. We will distinguish the cases when M has just one eigenvalue or only two. Let us notice that, by Proposition 6(2), to simplify the calculations, we may assume $\Lambda(M) = \{0\}$ and $\Lambda(M) = \{0, \alpha\}$, with $\alpha \neq 0$, respectively.

5.1 $\Lambda(M) = \{0\}$

In this subsection, we will assume that the minimal polynomial of M is λ^2. Since, by Proposition 4, we have

$$\Lambda^{(g)}_{\varepsilon,k}(M) = \{z \in \mathbb{C} : \sigma_{N-k+1}(zI_N - M) \leq \varepsilon\},$$

then the problem is reduced to calculate the singular values of $zI_N - M$. Now, since, by Remark 2, we may assume that M is in form (2), then given a $z \in \mathbb{C}$ it turns out that the singular values of $zI_N - M$ are

$$(|z| \; r \text{ times}) \bigcup_{i=1}^{q} \left(\sigma_1 \begin{pmatrix} z & -\sigma_i \\ 0 & z \end{pmatrix}, \sigma_2 \begin{pmatrix} z & -\sigma_i \\ 0 & z \end{pmatrix} \right).$$

Here $\cup$ is not the set union. We use the union of finite decreasing sequences of real numbers, as the sequence that has the whole of the components of these sequences, arranged in decreasing order. So, by denoting for $i = 1, 2, \ldots, q$,

$$\begin{cases} \eta_i(z) := \sqrt{\dfrac{2|z|^2 + \sigma_i^2 + \sigma_i\sqrt{4|z|^2 + \sigma_i^2}}{2}} \\ \mu_i(z) := \sqrt{\dfrac{2|z|^2 + \sigma_i^2 - \sigma_i\sqrt{4|z|^2 + \sigma_i^2}}{2}}, \end{cases} \tag{6}$$

we infer that the singular values of $zI_N - M$ are

$$|z|, r \text{ times}, \eta_i(z), \mu_i(z), \; i = 1, 2, \ldots, q.$$

Now, to determine the set $\Lambda^{(\mathrm{g})}_{\varepsilon,k}(M)$, is enough to arrange the singular values of $zI_N - M$. The next result solves the question.

Proposition 18. *With the previous notations, we conclude that the $q + r + q$ singular values of $zI_N - M$ are*

$$\eta_1(z) \geq \eta_2(z) \geq \cdots \geq \eta_q(z) \geq |z| = \cdots = |z| \geq \mu_q(z) \geq \mu_{q-1}(z) \geq \cdots \geq \mu_1(z).$$

Proof. Firstly, it is deduced from (6) that $\eta_q(z) \geq |z| \geq \mu_q(z)$. Since $\sigma_i \geq \sigma_{i+1}$, from (6), $\eta_i(z) \geq \eta_{i+1}(z)$.

Secondly, we deduce from (6) that

$$\mu_{i+1}(z) \geq \mu_i(z) \Leftrightarrow \sigma_{i+1}^2 - \sigma_{i+1}\sqrt{4|z|^2 + \sigma_{i+1}^2} \geq \sigma_i^2 - \sigma_i\sqrt{4|z|^2 + \sigma_i^2}.$$

Therefore, since $\sigma_i \geq \sigma_{i+1}$, Proposition 9 assures us that $\mu_{i+1}(z) \geq \mu_i(z)$.

□

With this proposition, we are ready to describe the sets $\Lambda^{(\mathrm{g})}_{\varepsilon,k}(M)$.

Theorem 19. *Let $M \in \mathbb{C}^{N\times N}$ with minimal polynomial equal to λ^2, such that $\nu(M) = q + r$ and $N = 2q + r$. Then, we have*

$$\Lambda^{(\mathrm{g})}_{\varepsilon,k}(M) = \begin{cases} \left\{ z \in \mathbb{C} : |z| \leq \sqrt{\varepsilon^2 + \varepsilon\sigma_k} \right\}, & \text{if } k = 1, \ldots, q, \\ \{ z \in \mathbb{C} : |z| \leq \varepsilon \}, & \text{if } k = q+1, \ldots, q+r, \\ \left\{ z \in \mathbb{C} : |z| \leq \sqrt{\varepsilon^2 - \varepsilon\sigma_{N-k+1}} \right\}, & \text{if } k = q+r+1, \ldots, N. \end{cases}$$

Remark 3. Let us notice that, for $k = q + r + 1, \ldots, N$,

$$\Lambda^{(\mathrm{g})}_{\varepsilon,k}(M) = \emptyset \Leftrightarrow \varepsilon < \sigma_{N-k+1}.$$

Proof. As a first step, let us assume that $1 \leq k \leq q$. Since, by Proposition 4,

$$\Lambda^{(g)}_{\varepsilon,k}(M) = \{z \in \mathbb{C} : \sigma_{N-k+1}(zI_N - M) \leq \varepsilon\}$$

and $\sigma_{N-k+1}(zI_N - M) = \mu_k(z)$ by Proposition 18, from the given expression for $\mu_k(z)$ in (6), we have

$$\Lambda^{(g)}_{\varepsilon,k}(M) = \left\{z \in \mathbb{C} : 2|z|^2 + \sigma_k^2 - \sigma_k\sqrt{4|z|^2 + \sigma_k^2} \leq 2\varepsilon^2\right\}.$$

In order to simplify the presentation, we will denote by $x := |z|^2$ and momentarily, $a := \sigma_k$. So, what we want to determine is the set

$$\left\{x \geq 0 : 2x + a^2 - a\sqrt{4x + a^2} \leq 2\varepsilon^2\right\}. \tag{7}$$

Let us notice that for $x = \varepsilon^2 + \varepsilon a$ the equality is attained. Therefore, since the function

$$2x + a^2 - a\sqrt{4x + a^2}$$

is strictly increasing on $[0, \infty)$ (Proposition 10), the set given in (7) is characterized by $x \leq \varepsilon^2 + \varepsilon a$. That is, $\Lambda^{(g)}_{\varepsilon,k}(M) = \{z \in \mathbb{C} : |z|^2 \leq \varepsilon^2 + \varepsilon\sigma_k\}$.

The case of $q + 1 \leq k \leq q + r$ is deduced immediately from Proposition 4. Finally let us suppose that $q + r + 1 \leq k \leq N$. From Proposition 18

$$\sigma_j(zI_N - M) = \eta_j(z)$$

for $j = 1, \ldots, q$. Since $q + r + 1 \leq k \leq 2q + r$ we see that $1 \leq N - k + 1 \leq q$, thus $\sigma_{N-k+1}(zI_N - M) = \eta_{N-k+1}(z)$. From Proposition 4 and (6) we have

$$\Lambda^{(g)}_{\varepsilon,k}(M) = \left\{z \in \mathbb{C} : 2|z|^2 + \sigma_{N-k+1}^2 + \sigma_{N-k+1}\sqrt{4|z|^2 + \sigma_{N-k+1}^2} \leq 2\varepsilon^2\right\}.$$

Once again, in order to simplify the presentation, we will denote for $x := |z|^2$ and momentarily, $a := \sigma_{N-k+1}$. So, what we want to determine is the set

$$\{x \geq 0 : 2x + a^2 + a\sqrt{4x + a^2} \leq 2\varepsilon^2\}. \tag{8}$$

Let us observe that if $\varepsilon < a$, then, since $x \geq 0$, the set (8) is empty. Now let us notice that for $x = \varepsilon^2 - \varepsilon a$ equality holds. Therefore, as the function

$$2x + a^2 + a\sqrt{4x + a^2}$$

is strictly increasing on $[0, \infty)$, the set given in (8) is $\{x \geq 0 \colon x \leq \varepsilon^2 - \varepsilon a\}$. That is, $\Lambda^{(g)}_{\varepsilon,k}(M) = \{z \in \mathbb{C} : |z|^2 \leq \varepsilon^2 - \varepsilon\sigma_{N-k+1}\}$.

□

Let us recall that the *index of an eigenvalue* λ_0 of a square complex matrix M is the multiplicity of λ_0 as a root of the minimal polynomial of M.

Corollary 20. *Let $M \in \mathbb{C}^{N\times N}$ with $\Lambda(M) = \{\lambda_0\}$, such that the index of λ_0 is 2, and $\nu(\lambda_0 I_N - M) = q + r$ where $N = 2q + r$. Then, we have*

$$\Lambda^{(g)}_{\varepsilon,k}(M) = \begin{cases} \left\{z \in \mathbb{C} : |z - \lambda_0| \leq \sqrt{\varepsilon^2 + \varepsilon\sigma_k(\lambda_0 I_N - M)}\right\}, & \text{if } k = 1, \ldots, q, \\ \{z \in \mathbb{C} : |z - \lambda_0| \leq \varepsilon\}, & \text{if } k = q + 1, \ldots, q + r, \\ \left\{z \in \mathbb{C} : |z - \lambda_0| \leq \sqrt{\varepsilon^2 - \varepsilon\sigma_{N-k+1}(\lambda_0 I_N - M)}\right\}, & \\ & \text{if } k = q + r + 1, \ldots, N. \end{cases}$$

5.2 $\Lambda(M) = \{0, \alpha\}$

Now we will analyze the sets $\Lambda^{(g)}_{\varepsilon,k}(M)$, assuming that the minimal polynomial of M is equal to $\lambda(\lambda-\alpha)$, with $\alpha \neq 0$. By using a similar reasoning to that used in previous subsection, and assuming that M is in the form (3), if we denote by

$$f_i(z) := s_i^2 + |z-\alpha|^2 + |z|^2, \tag{9}$$

we deduce that the singular values of $zI_N - M$ are

$$|z|\, r \text{ times}, \ \eta_i(z), \mu_i(z), \ i = 1, 2, \dots, q,$$

where, for $i = 1, 2, \dots, q$,

$$\begin{cases} \eta_i(z) := \sqrt{\dfrac{f_i(z) + \sqrt{f_i(z)^2 - (2|z-\alpha|\,|z|)^2}}{2}}, \\ \mu_i(z) := \sqrt{\dfrac{f_i(z) - \sqrt{f_i(z)^2 - (2|z-\alpha|\,|z|)^2}}{2}}. \end{cases} \tag{10}$$

Now, to determine the set $\Lambda^{(g)}_{\varepsilon,k}(M)$, it is enough to arrange the singular values of $zI_N - M$ in decreasing order. The next result solves our question.

Proposition 21. *We infer that the $q + r + q$ singular values of $zI_N - M$ are*

$$\eta_1(z) \geq \eta_2(z) \geq \cdots \geq \eta_q(z) \geq |z| = \cdots = |z| \geq \mu_q(z) \geq \mu_{q-1}(z) \geq \cdots \geq \mu_1(z).$$

Proof. In the first place, $\eta_i(z) \geq \eta_{i+1}(z)$ follows immediately from (10) because the function $x + \sqrt{x^2 - a}$, where $a \geq 0$, is increasing on $[0, \infty)$. On the other hand, $\eta_i(z) \geq |z|$ if and only if $2\eta_i(z)^2 \geq 2|z|^2$. But from (10) we have

$$\begin{aligned} 2\eta_i(z)^2 \geq\ & |z|^2 + |z-\alpha|^2 + \sqrt{(|z|^2 - |z-\alpha|^2)^2} \\ =\ & |z|^2 + |z-\alpha|^2 + \big||z|^2 - |z-\alpha|^2\big| = 2\max\{|z|^2, |z-\alpha|^2\} \\ \geq\ & 2|z|^2. \end{aligned}$$

To prove that $\mu_i(z) \leq \mu_{i+1}(z) \Leftrightarrow 2\mu_i(z)^2 \leq 2\mu_{i+1}(z)^2$, since $s_i \geq s_{i+1}$, from (10) it is enough to show that the function

$$g_z(x) := x^2 - \sqrt{(x^2 + |z-\alpha|^2 + |z|^2)^2 - (2|z-\alpha|\,|z|)^2}$$

is decreasing on $[0, \infty)$. This fact is guaranteed by Proposition 11.

Finally, $\mu_i(z) \leq |z|$ if and only if $2\mu_i(z)^2 \leq 2|z|^2$, and, as well, $\mu_i(z) \leq \mu_{i+1}(z)$, then from (10) we conclude that

$$\begin{aligned} 2\mu_i(z)^2 \leq\ & |z|^2 + |z-\alpha|^2 - \sqrt{(|z|^2 - |z-\alpha|^2)^2} \\ =\ & |z|^2 + |z-\alpha|^2 - \big||z|^2 - |z-\alpha|^2\big| = 2\min\{|z|^2, |z-\alpha|^2\} \\ \leq\ & 2|z|^2. \end{aligned}$$

□

As a consequence, for the sets $\Lambda^{(g)}_{\varepsilon,k}(M)$ we deduce the next result.

Theorem 22. *Let $M \in \mathbb{C}^{N\times N}$ with minimal polynomial equal to $\lambda(\lambda-\alpha)$, with $\alpha \neq 0$, such that $\nu(\alpha I_N - M) = q$ and $\nu(M) = q + r$. Let $h(z) := |z-\alpha|\,|z|$ and with the notations in* (3) *let $f_i(z) := s_i^2 + |z-\alpha|^2 + |z|^2$. Then we have*

(1) *If* $k = 1, 2, \ldots, q$,

$$\Lambda^{(\mathrm{g})}_{\varepsilon,k}(M) = \{z \in \mathbb{C} : \varepsilon^4 + h(z)^2 \le \varepsilon^2 f_k(z)\} \cup \{z \in \mathbb{C} : f_k(z) \le 2\varepsilon^2\}.$$

(2) *For* $k = q+1, \ldots, q+r$, $\Lambda^{(\mathrm{g})}_{\varepsilon,k}(M) = \{z \in \mathbb{C} : |z| \le \varepsilon\}$.

(3) *If* $k = q+r+1, \ldots, N$,

$$\Lambda^{(\mathrm{g})}_{\varepsilon,k}(M) = \{z \in \mathbb{C} : \varepsilon^2 f_{N-k+1}(z) \le \varepsilon^4 + h(z)^2\} \cap \{z \in \mathbb{C} : f_{N-k+1}(z) \le 2\varepsilon^2\}.$$

Proof. (1) Following the reasonings used in the proof of Theorem 19, from (10) and Proposition 21, for $k = 1, 2, \ldots, q$, we deduce that

$$\begin{aligned} \Lambda^{(\mathrm{g})}_{\varepsilon,k}(M) &= \{z \in \mathbb{C} : f_k(z) - \sqrt{f_k(z)^2 - 4h(z)^2} \le 2\varepsilon^2\} \\ &= \{z \in \mathbb{C} : f_k(z) - 2\varepsilon^2 \le \sqrt{f_k(z)^2 - 4h(z)^2}\}. \end{aligned} \tag{11}$$

Now let us observe that if

$$f_k(z) \le 2\varepsilon^2 \Rightarrow z \in \Lambda^{(\mathrm{g})}_{\varepsilon,k}(M). \tag{12}$$

Now let us suppose $f_k(z) > 2\varepsilon^2$. Then, from (11) we find that

$$z \in \Lambda^{(\mathrm{g})}_{\varepsilon,k}(M) \Leftrightarrow f_k(z)^2 + 4\varepsilon^4 - 4\varepsilon^2 f_k(z) \le f_k(z)^2 - 4h(z)^2 \Leftrightarrow \varepsilon^4 + h(z)^2 \le \varepsilon^2 f_k(z),$$

this together with (12) in (11) proves the case (1). The case (2) is deduced immediately from Proposition 21. Finally, for the case (3), from (10) and Proposition 21, for $k = q+r+1, \ldots, N$, we have

$$\begin{aligned} \Lambda^{(\mathrm{g})}_{\varepsilon,k}(M) &= \{z \in \mathbb{C} : f_{N-k+1}(z) + \sqrt{f_{N-k+1}(z)^2 - 4h(z)^2} \le 2\varepsilon^2\} \\ &= \{z \in \mathbb{C} : \sqrt{f_{N-k+1}(z)^2 - 4h(z)^2} \le 2\varepsilon^2 - f_{N-k+1}(z)\}. \end{aligned}$$

Let us notice that if $2\varepsilon^2 < f_{N-k+1}(z)$, then $\Lambda^{(\mathrm{g})}_{\varepsilon,k}(M) = \emptyset$. Therefore, assuming that $2\varepsilon^2 \ge f_{N-k+1}(z)$, we infer that

$$z \in \Lambda^{(\mathrm{g})}_{\varepsilon,k}(M) \Leftrightarrow f_{N-k+1}(z)^2 - 4h(z)^2 \le 4\varepsilon^4 + f_{N-k+1}(z)^2 - 4\varepsilon^2 f_{N-k+1}(z).$$

This proves (3).

□

6 Conclusions

Let A, B be square complex matrices of orders n and m, respectively. Let us assume that $n \ge m$ and that the chain of invariant factors of B contains t nontrivial factors (counting repetitions). Let us assume that for each $\varepsilon > 0$ and k the matrices A and B have the same ε-pseudospectrum of geometric multiplicity $\ge k$. We have shown that the last t elements of the chain of invariant factors of

A are equal to those of B. Thus, when the sizes of A and B are equal, theses conditions imply that the matrices A and B are similar.

It is well known that if $A, B \in \mathbb{C}^{n\times n}$ are unitarily similar, then for every $\varepsilon > 0$ and $k = 1, \ldots, n$ the pseudospectra $\Lambda^{(g)}_{\varepsilon,k}(A)$ and $\Lambda^{(g)}_{\varepsilon,k}(B)$ are equal. This lead us to analyze to what extent the pseudospectra $\Lambda^{(g)}_{\varepsilon,k}(M)$ of any square complex matrix M are determined by the invariant factors of M. We have managed to describe these pseudospectra for the matrices M with quadratic minimal polynomial. In particular, if $\Lambda(M) = \{\lambda_0\}$ and the index of λ_0 is two, then these pseudospectra are closed disks whose radii depend on the singular values of $\lambda_0 I - M$.

References

[1] M. Fortier Bourque, T. Ransford. Super-identical pseudospectra, *J. London Math. Soc.* (2) 79 (2009) 511–528.

[2] A. George, K. D. Ikramov. Unitary similarity of matrices with quadratic minimal polynomials. *Linear Algebra Appl.*, 349: (2002)11–16.

[3] G.H. Hardy. *Orders of infinity.* Hafner Publishing Company, New York, 1971.

[4] L. Hogben. *Handbook of Linear Algebra.* Chapman, Hall/CRC, 2007.

[5] M. Karow. *Geometry of spectral value sets.* Ph.D. Thesis. University of Bremen, 2003.

[6] J. F. Queiró, E. Marques de Sá. Singular values and invariant factors of matrix sums and products. *Linear Algebra Appl.*, 225 (1995)43–56.