My road to spectral perturbation of matrices

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Outline

The set $\{M \in \mathbb{C}^{n\times n} : \#\Lambda(M) = n\}$ **is open and dense, 1972→**

Bifurcation points of the rank of a matrix function

Matrix functions which commute with their derivative, 1975→

Singular values, 1981→

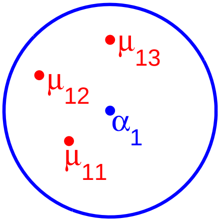
Perturbation of Jordan form, 1983→

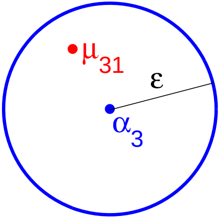
Sylvester equation $AX - XB = C$**, 1975→**

Smooth jordanization of matrix functions, 1986→

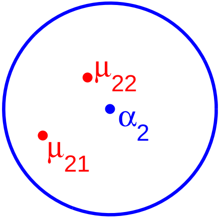
Nearest matrices. Pseudospectra, 1992–1994→

The set $\{M \in \mathbb{C}^{n\times n} : \#\Lambda(M) = n\}$ is open and dense, 1972→





$$\sum_{\mu\in\Lambda(A')\cap B(\alpha,\varepsilon)} \mathrm{m}(\mu, A') = \mathrm{m}(\alpha, A), \quad \forall \alpha \in \Lambda(A).$$



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Bifurcation points of the rank of $A(t)$

$$A\colon (\alpha,\beta) \to \mathbb{C}^{m\times n}, \qquad \text{continuous.}$$

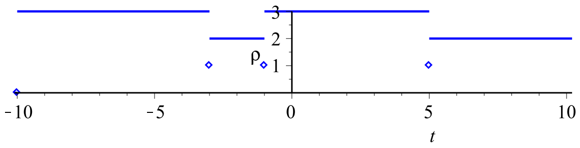
Bifurcation point of $\rho(t) := \operatorname{rank} A(t) \iff$ **discontinuity point of** ρ.

$$\mathcal{B}, \text{ closed, } \mathring{\mathcal{B}} = \varnothing.$$

$$(\alpha,\beta) \setminus \mathcal{B} = \dot{\bigcup_{k=1}^{\infty}} (\alpha_k, \beta_k), \qquad \rho(t) \text{ constant on } (\alpha_k,\beta_k).$$

⚠ $\mathcal{B}$ can be uncountable: Cantor set.

$$\overline{\dot{\bigcup_{k=1}^{\infty}} (\alpha_k, \beta_k)} = [\alpha, \beta].$$



Segre bifurcation

2×2 matrix functions which commute with their derivative, 1975→

$$B(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \quad B'(t)B(t) = B(t)B'(t). \quad \Longrightarrow$$

$$\Omega := \{t : b(t) \neq 0 \vee c(t) \neq 0 \vee a(t) \neq d(t)\} \text{ open},$$

$$\Omega = \dot{\bigcup_{k=1}^{\infty}} (\alpha_k, \beta_k), \quad C := (\alpha, \beta) \setminus \Omega \text{ closed in } (\alpha, \beta).$$

$$B(t) = \begin{pmatrix} a(t) & \lambda_k f_k(t) \\ \mu_k f_k(t) & a(t) + \nu_k f_k(t) \end{pmatrix} \approx \begin{cases} \begin{pmatrix} \lambda_1(t) & 0 \\ 0 & \lambda_2(t) \end{pmatrix}, & \text{if } \nu_k^2 + 4\lambda_k \mu_k \neq 0, \\ \begin{pmatrix} \lambda_1(t) & 1 \\ 0 & \lambda_1(t) \end{pmatrix}, & \text{if } \nu_k^2 + 4\lambda_k \mu_k = 0, \end{cases}$$

$$B(t) = \begin{pmatrix} a(t) & 0 \\ 0 & a(t) \end{pmatrix}, t \in C;$$

constant Segre characteristic: $[(1),(1)]$ or $[(2)]$ on (α_k, β_k), and $[(1,1)]$ on C.

Rodríguez-Cano

Smooth jordanization of matrix functions with constant Segre characteristic, 1975→

Bogdanov and Chebotarev (1959)

$A\colon(\alpha,\beta)\to\mathbb{C}^{n\times n}$, $A\in C^p$. $\Longrightarrow$ $\exists$ functions $\lambda_j\colon(\alpha,\beta)\to\mathbb{C}$ $(j=1,\dots,q)$,
$P\colon(\alpha,\beta)\to\mathbb{C}^{n\times n}$ of class C^p s.t. $\forall t\in(\alpha,\beta)$,

$$\Lambda(A(t))=\{\lambda_1(t),\dots,\lambda_q(t)\},\quad P(t)^{-1}A(t)P(t)=J(t),\ \text{Jordan form.}$$

Singular values, 1981→

Figure: University of Coimbra, Portugal

Distance to the set of matrices with lower rank

$\operatorname{rank} M = r \iff \sigma_1(M) \geq \cdots \geq \sigma_r(M) > 0 = \sigma_{r+1}(M) = \cdots = \sigma_p(M)$.

If $0 \leq k < r$

$$\min_{\substack{X \in \mathbb{C}^{m \times n} \\ \operatorname{rank} X \leq k}} \|X - M\| = \sigma_{k+1}(M).$$

Sodupe, G.

Segre and Weyr partitions of an eigenvalue α, 1983→

$$P^{-1}AP = \begin{pmatrix} \ddots & & & \\ & \boxed{\begin{matrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & \alpha \end{matrix}} & & \\ & & \boxed{\begin{matrix} \alpha & 1 \\ 0 & \alpha \end{matrix}} & \\ & & & \boxed{\begin{matrix} \alpha & 1 \\ 0 & \alpha \end{matrix}} \\ & & & & \ddots \end{pmatrix}$$

$$\begin{array}{ccccc} 4 & \bullet & \bullet & \bullet & \bullet \\ 2 & \bullet & \bullet & & \\ 2 & \bullet & \bullet & & \\ & 3 & 3 & 1 & 1 \end{array}$$

$$s(\alpha, A) = (4, 2, 2)$$

$$w(\alpha, A) = (3, 3, 1, 1) =: \overline{(4, 2, 2)}$$ conjugate partition

$(5, 5, 3, 2) \cup (4, 3, 2, 1, 1) := (5, 5, 4, 3, 3, 2, 2, 1, 1)$

Majorization: $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p \geq 0, \qquad \nu_1 \geq \nu_2 \geq \cdots \geq \nu_p \geq 0,$

$$\mu \prec \nu$$

if $\quad \mu_1 + \cdots + \mu_k \leq \nu_1 + \cdots + \nu_k, \qquad \sum_{i=1}^{p} \mu_i = \sum_{i=1}^{p} \nu_i.$

Perturbation of the Jordan form, 1983→

Necessary conditions: Markus and Parilis, ... $A \in \mathbb{C}^{n\times n}$, $\varepsilon > 0$ adequate to A. Then $\exists r > 0$ s.t. $\forall A' \in B(A,r) \subset \mathbb{C}^{n\times n}$:

(i)

$$\Lambda(A') \subset \bigcup_{\alpha\in\Lambda(A)} B(\alpha,\varepsilon),$$

(ii)

$$\bigcup_{\mu\in\Lambda(A')\cap B(\alpha,\varepsilon)} \mathrm{w}(\mu,A') \prec \mathrm{w}(\alpha,A), \quad \forall\alpha \in \Lambda(A).$$

Perturbation of the Jordan form, 1983→

Sufficient conditions: Markus and Parilis, . . . $A \in \mathbb{C}^{n\times n}$, $\varepsilon > 0$ adequate to A. $\forall\alpha \in \Lambda(A)$ let $t_\alpha \in \mathbb{N}^*$ and let $b_{\alpha 1}, \dots, b_{\alpha t_\alpha}$ be non-null partitions. Then $\forall\delta > 0, \exists A'$ s.t. $\|A' - A\| < \delta$ and

(i)

$$\Lambda(A') \subset \bigcup_{\alpha\in\Lambda(A)} B(\alpha,\varepsilon),$$

(ii) $\forall\alpha \in \Lambda(A)$ the matrix A' has precisely t_α eigenvalues $\mu_{\alpha 1}, \dots, \mu_{\alpha t_\alpha}$ in $B(\alpha,\varepsilon)$ and

$$b_{\alpha j} = \mathrm{w}(\mu_{\alpha j}, A') \quad (j = 1, \dots, t_\alpha),$$

if and only if

$$\bigcup_{j=1}^{t_\alpha} b_{\alpha j} \prec \mathrm{w}(\alpha, A), \quad \forall\alpha \in \Lambda(A).$$

Perturbation of the canonical form of Brunovsky **de Hoyos, Zaballa, G., Baragaña, Beitia.** Perturbation of the canonical form of Kronecker Pokrzywa (1986),. . . **de Hoyos, G.** Stability of invariant subspaces **Velasco, G.**

Sylvester equation $AX - XB = C$, 1975→

$\nu(A,B) := \dim\{X \in \mathbb{C}^{m\times n} : AX - XB = O\}$. If $\Lambda(A) \cap \Lambda(B) = \{\lambda_1, \dots, \lambda_s\}$,

$$\nu(A,B) = \sum_{i=1}^{s} \mathrm{w}(\lambda_i, A) \cdot \mathrm{w}(\lambda_i, B).$$

$$A \approx B \iff \nu(A,A) = \nu(A,B) = \nu(B,B) \iff$$

$$\mathrm{rank}(A \otimes I_n - I_n \otimes A) = \mathrm{rank}(A \otimes I_n - I_m \otimes B) = \mathrm{rank}(B \otimes I_m - I_m \otimes B).$$

Feedback equivalence of matrix pairs $(A,B) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m}$.
Strict equivalence of rectangular matrix pencils $\lambda F - G$.

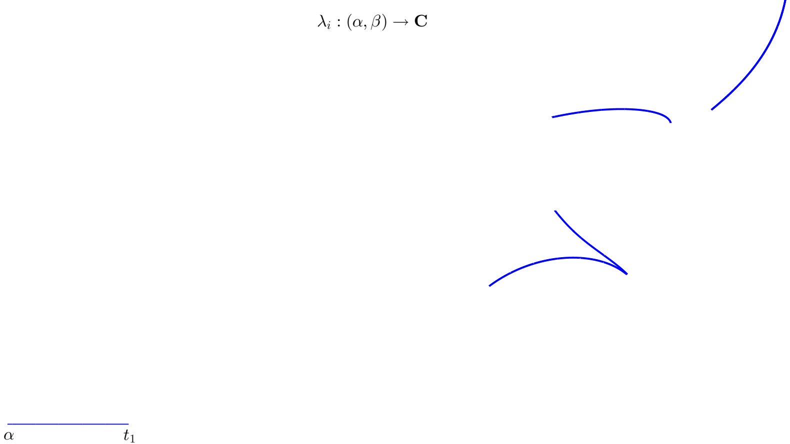
Beitia, G.

Ortiz de Elguea

Bifurcation points of the spectrum of $A(t)$

$$\Lambda(A(t)) = \{\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), \lambda_5(t)\}$$

$$\lambda_i : (\alpha, \beta) \to \mathbf{C}$$



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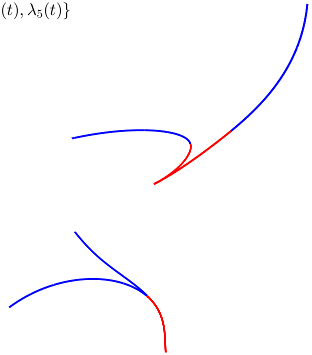

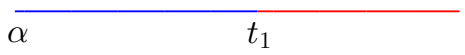

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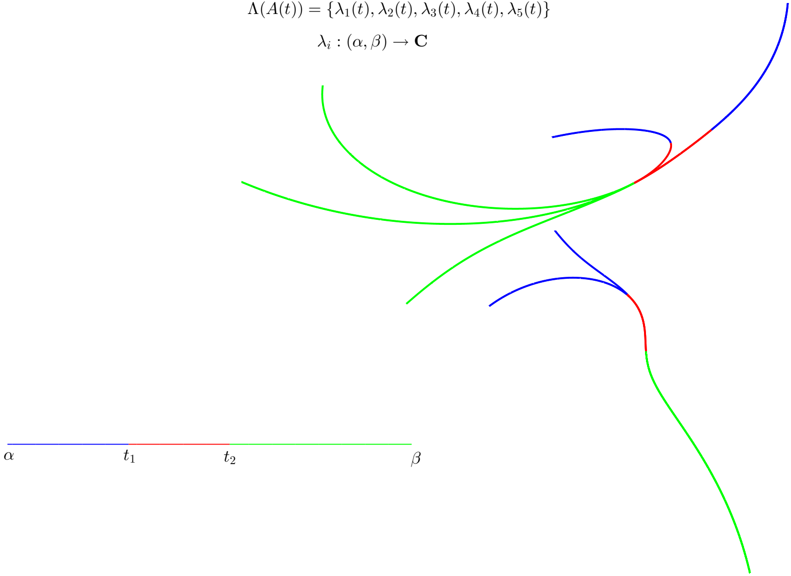

Bifurcation points of the Segre characteristic $\mathrm{s}(A(t))$

$A:(\alpha,\beta)\to\mathbb{C}^{n\times n}$ continuous, $r:=\max_{t\in(\alpha,\beta)}\#\Lambda\big(A(t)\big)$ $\Longrightarrow$
$\exists\lambda_1,\dots,\lambda_r:(\alpha,\beta)\to\mathbb{C}$ continuous s.t. $\forall t\in(\alpha,\beta)$,

$$\Lambda(A(t))=\{\lambda_1(t),\dots,\lambda_r(t)\}.$$

Bifurcation points of the spectrum of A.

$p(\lambda,t)=\det(\lambda I_n-A(t))$, $N(t):=\#\Lambda(A(t))$,

$$N(t)=\operatorname{rank}R\big(p(\lambda,t),p'_\lambda(\lambda,t)\big)-n+1.$$

Bifurcation points of a Segre partition of A. rank bifurcation

$$\overline{\mathrm{s}(\lambda_i(t),A(t))}=\mathrm{w}(\lambda_i(t),A(t))=\big(m_{i1}(t),m_{i2}(t),\dots,m_{i\ell}(t),\dots\big)$$

$$m_{i1}(t)+m_{i2}(t)+\cdots+m_{i\ell}(t)=\dim\operatorname{Ker}\big(\lambda_i(t)I_n-A(t)\big)^\ell$$

$$\overline{\dot{\bigcup_{k=1}^{\infty}}(\alpha_k,\beta_k)}=[\alpha,\beta].$$

⚠ The α_k and the β_k are bifurcation points of $\mathrm{s}(A(t))$. There can be more.

$$A(t):=\begin{pmatrix} te^{2it} & f(t)\\ 0 & te^{2it}\end{pmatrix},\quad f(t):=\begin{cases}e^{-1/t}, & t>0,\\ 0, & t\le 0;\end{cases}\qquad \mathrm{s}(te^{2it},A(t))=\begin{cases}(2), & t>0,\\ (1,1), & t\le 0.\end{cases}$$

Bifurcation points of the Segre characteristic $\mathrm{s}(A(t))$

$A : (\alpha, \beta) \to \mathbb{C}^{n\times n}$ continuous, $r := \max_{t\in(\alpha,\beta)} \#\Lambda\big(A(t)\big) \implies$
$\exists \lambda_1, \dots, \lambda_r : (\alpha, \beta) \to \mathbb{C}$ continuous s.t. $\forall t \in (\alpha, \beta)$,

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$p(\lambda, t) = \det(\lambda I_n - A(t))$, $\quad N(t) := \#\Lambda(A(t))$,

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Bifurcation points of a Segre partition of A.

rank bifurcation

$$\overline{\mathrm{s}(\lambda_i(t), A(t))} = \mathrm{w}(\lambda_i(t), A(t)) = \big(m_{i1}(t), m_{i2}(t), \dots, m_{i\ell}(t), \dots\big)$$

$$m_{i1}(t) + m_{i2}(t) + \cdots + m_{i\ell}(t) = \dim \operatorname{Ker}\big(\lambda_i(t) I_n - A(t)\big)^\ell$$

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Smooth jordanization of matrix functions $A : (\alpha,\beta) \to \mathbb{C}^{n\times n}$

$A \in C^p$, $r := \max_{t\in(\alpha,\beta)} \#\Lambda\big(A(t)\big)$. Isolated bifurcation points.

$\exists P : (\alpha,\beta) \to \mathbb{C}^{n\times n}$, $P \in C^p$ s.t.

$$P(t)^{-1}A(t)P(t) = J(t)$$

$$\Updownarrow$$

1. the union of Segre partitions

$$\bigcup_{z\in\Lambda(A(t))} s(z, A(t))$$

   is constant on (α,β);

2. $\exists \lambda_1, \ldots, \lambda_r : (a,b) \to \mathbb{C}$ of class C^p s.t.:

   2.1 $\forall t \in (\alpha,\beta)$, $\quad \Lambda(A(t)) = \{\lambda_1(t), \ldots, \lambda_r(t)\}$;

   2.2 for each *bifurcation point* t_0 of $s(A(t))$, let

$$\{i_1, \ldots, i_u\}, \{j_1, \ldots, j_v\} \subset \{1, \ldots, r\}$$

   associated to t_0; then the sums

$$\dotplus_{k=1}^{u} \lim_{t\to t_0^-} \operatorname{Ker}(\lambda_{i_k}(t)I_n - A(t))^n \quad \text{and} \quad \dotplus_{k=1}^{v} \lim_{t\to t_0^+} \operatorname{Ker}(\lambda_{j_k}(t)I_n - A(t))^n$$

   are direct and $= \mathbb{C}^n$.

Thijsee (1985), Evard, Velasco, G.

Wilkinson's problem, 1992→

◂ simple spectra

- $A \in \mathbb{C}^{n\times n}, z_0 \in \mathbb{C}$,

$$\min_{\substack{X\in\mathbb{C}^{n\times n}\\ \mathrm{m}(z_0,X)\geq 2}} \|X-A\| = \max_{t\in\mathbb{R}} \sigma_{2n-1}\begin{pmatrix} z_0 I_n - A & tI_n \\ O & z_0 I_n - A \end{pmatrix} =: h_2(z_0).$$

$$\Lambda_2(X) := \{\xi \in \Lambda(X) : \mathrm{m}(\xi, X) \geq 2\}$$

$$\min_{\Lambda_2(X)\neq\emptyset} \|X-A\| = \min_{z\in\mathbb{C}} h_2(z).$$

Malyshev (1999)

- Let $z_0 \in \mathbb{C}$. Then

$$\min_{\substack{X\in\mathbb{C}^{n\times n}\\ \mathrm{m}(z_0,X)\geq 3}} \|X-A\| = \max_{(t_1,t_2,t_3,t_4)\in\mathbb{R}^4} \sigma_{3n-2}\begin{pmatrix} z_0 I_n - A & t_1 I_n & (t_3+t_4\mathrm{i})I_n \\ O & z_0 I_n - A & t_2 I_n \\ O & O & z_0 I_n - A \end{pmatrix} =: h_3(z_0).$$

Ikramov and Nazari

Armentia, González de Durana, de Hoyos, G., Velasco.

Wilkinson's problem, 1992→

◂ simple spectra

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$$\Lambda_2(X) := \{\xi \in \Lambda(X) : \mathrm{m}(\xi, X) \geq 2\}$$

$$\min_{\Lambda_2(X)\neq\varnothing} \|X - A\| = \min_{z\in\mathbb{C}} h_2(z).$$

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Pseudospectra, 1994→

$\delta \geq 0$

$$\Lambda_\delta(A) = \Lambda_{\delta,1}(A) := \bigcup_{\|X-A\|\leq\delta} \Lambda(X),$$

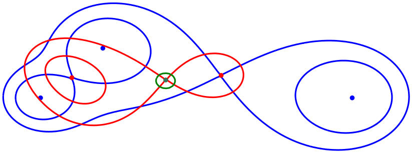
$$\Lambda_{\delta,2}(A) := \bigcup_{\|X-A\|\leq\delta} \Lambda_2(X),$$

$$\Lambda_{\delta,3}(A) := \bigcup_{\|X-A\|\leq\delta} \Lambda_3(X).$$

$$\Lambda_\delta(A) = \{z \in \mathbb{C} : h_1(z) := \sigma_n(zI_n - A) \leq \delta\},$$

$$\Lambda_{\delta,2}(A) = \{z \in \mathbb{C} : h_2(z) \leq \delta\},$$

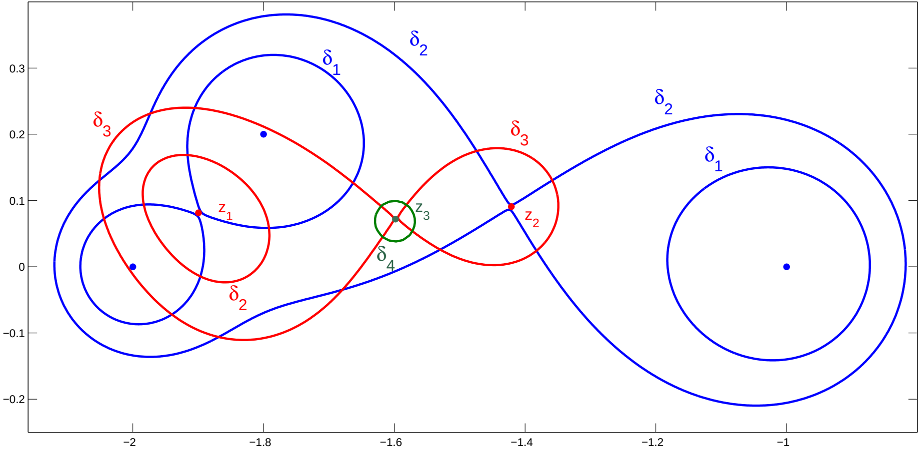
$$\Lambda_{\delta,3}(A) = \{z \in \mathbb{C} : h_3(z) \leq \delta\}.$$



Armentia, Velasco, G.

Wilkinson's problem and pseudospectra, 1999→

$$A = \begin{pmatrix} -1 & 5 & 6 \\ 0 & -2 & 0 \\ 0 & 0 & -1.8+0.2i \end{pmatrix}, \quad \delta_1 < \delta_2 < \delta_3 < \delta_4.$$



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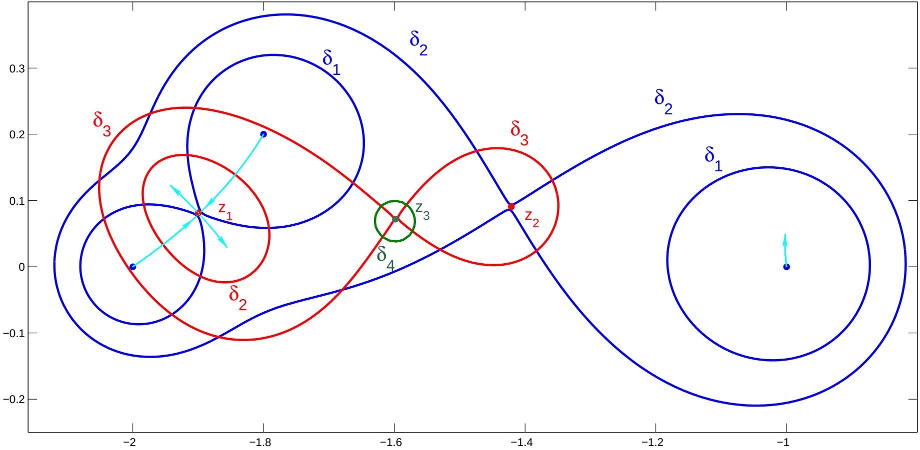
$$A_1 := A + \sigma_n (z_1 I_n - A) u_n v_n^*, \quad X_1(t) := (1-t)A + tA_1, \; 0 \le t \le 1.2$$



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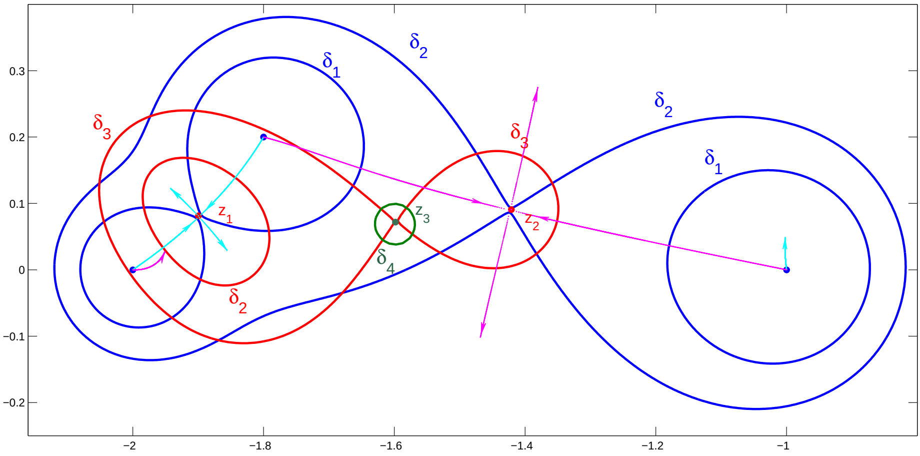
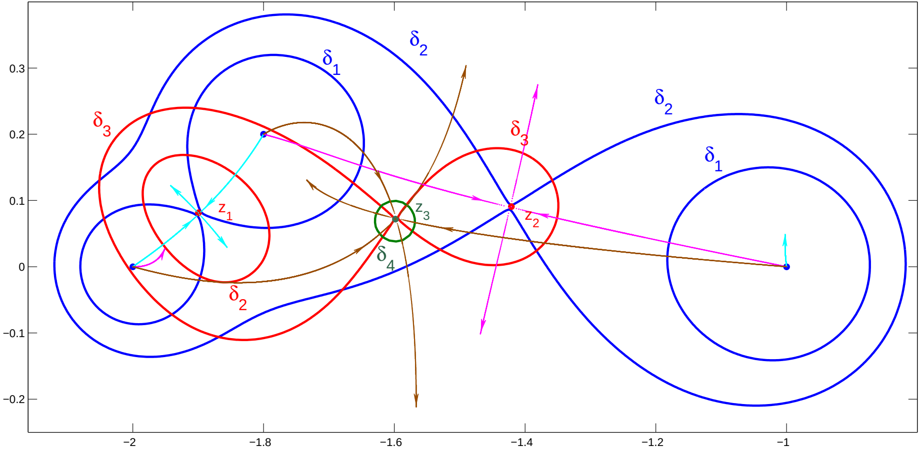
$$A_2 := A + \sigma_n (z_2 I_n - A) u_n v_n^*, \quad X_2(t) := (1-t)A + tA_2,\ 0 \le t \le 1.2$$



Wilkinson's problem and pseudospectra, 1999→

$$A = \begin{pmatrix} -1 & 5 & 6 \\ 0 & -2 & 0 \\ 0 & 0 & -1.8+0.2i \end{pmatrix}, \quad \delta_1 < \delta_2 < \delta_3 < \delta_4.$$

$$A_3 := A + \sigma_{2n-1} \begin{pmatrix} z_3 I_n - A & t_0 I_n \\ O & z_3 I_n - A \end{pmatrix} [u_1, u_2][v_1, v_2]^\dagger, \quad X_3(t) := (1-t)A + tA_3, \; 0 \le t \le 1.2$$



Conclusion

spectral perturbation = rank + Weyr

Thanks for your attendance!