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Geometric multiplicity margin for a submatrix

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Abstract: Let G be a square complex matrix with less than k nonconstant invariant polynomials. We find a complex matrix that gives an optimal approximation to G among all possible matrices that have more than or equal to k invariant polynomials, obtained by varying only the entries of a bottom right submatrix of G.

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1. Introduction

Notations

In this paper we use the following notation. By $\mathbb{C}$ we denote the field of complex numbers, and $\mathbb{C}^{m\times n}$ the set of $m \times n$ matrices with entries in $\mathbb{C}$. We always will use the spectral norm over $\mathbb{C}^{p\times q}$

$$\|M\| = \max_{\substack{x \in \mathbb{C}^{q\times 1} \\ \|x\|_2 = 1}} \|Mx\|_2, \; M \in \mathbb{C}^{p\times q}.$$

The singular values of a matrix M are denoted by $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_k(M)$, where $k = \min(p, q)$. It is well known that $\|M\| = \sigma_1(M)$. The Moore-Penrose inverse of M is denoted by $M^\dagger$ and M^* denotes the conjugate transpose of M. And, when $p = q$, we denote by $\Lambda(M)$ the spectrum or set of distinct eigenvalues of M.

Let $A \in \mathbb{C}^{n\times n}$; the geometric multiplicity of an eigenvalue λ_0 of A is the number of Jordan blocks associated to λ_0 into the Jordan canonical form of A; we denote this number by $\text{gm}(\lambda_0, A)$. So $\text{gm}(\lambda_0, A)$ is the maximum number of linearly independent eigenvectors of A associated to λ_0; this implies that

$$\mathrm{gm}(\lambda_0, A) = \dim \mathrm{Ker}(\lambda_0 I_n - A).$$

Let k, $2 \leq k \leq n$, be an integer. A complex number λ_0 is called a *k-derogatory eigenvalue* of a matrix $A \in \mathbb{C}^{n\times n}$ if $\mathrm{gm}(\lambda_0, A) \geq k$. We will say that a matrix $A \in \mathbb{C}^{n\times n}$ is k-derogatory if A has a k-derogatory eigenvalue. We will denote by $i(A)$ the number of nonconstant (or nontrivial) invariant factors of A. It can be observed that $i(A)$ is the greatest geometric multiplicity of the eigenvalues of A.

We denote by $\mathcal{M}_k \subset \mathbb{C}^{n\times n}$ the set

$$\mathcal{M}_k := \{A \in \mathbb{C}^{n\times n} : i(A) < k\}.$$

That is to say, $\mathcal{M}_k$ is the set of the matrices A with all its eigenvalues with geometric multiplicity $< k$. Thus, in particular, $\mathcal{M}_2$ is the set of $n \times n$ nonderogatory matrices. Since

$$A \in \mathcal{M}_k \Leftrightarrow \text{ for all } \lambda \in \Lambda(A) \quad \mathrm{rank}(\lambda I - A) > n - k,$$

the set $\mathcal{M}_k$ is open. So, its complementary set $\mathcal{M}_k^c$ is closed. Then, given a matrix $\mathsf{D} \in \mathcal{M}_k$, if we consider a closed ball $\overline{B}(\mathsf{D}, \rho) \subset \mathbb{C}^{n\times n}$, with center at

D and radius ρ, it makes sense to find the distance from D to the compact set $\mathcal{M}_k^c \cap \overline{B}(\mathsf{D}, \rho)$ of k-derogatory matrices in the ball.

Antecedent of the problem

The problem of finding

$$\min\{\|\mathsf{Y} - \mathsf{D}\| \ : \ \mathsf{Y} \in \mathcal{M}_k^c\}$$

was addressed in [7, Theorem 4.1]. There its authors calculated this minimum value and also the matrix where it is attained. They obtained the formula

$$\min_{\substack{\mathsf{Y} \in \mathbb{C}^{n\times n} \\ i(\mathsf{Y}) \geq k}} \|\mathsf{Y} - \mathsf{D}\| = \min_{\lambda \in \mathbb{C}} \sigma_{n-(k-1)}(\lambda I_n - \mathsf{D}) \tag{1.1}$$

for the minimum and also proved that if $\lambda_0 \in \mathbb{C}$ is a point where the function $\lambda \mapsto \sigma_{n-(k-1)}(\lambda I_n - \mathsf{D})$ attains its minimum value, then a matrix Y_1 where the minimum of the left hand side of (1.1) is reached is given by

$$\mathsf{Y}_1 = \mathsf{D} + s_{n-(k-1)} u_{n-(k-1)} v_{n-(k-1)}^* + \cdots + s_n u_n v_n^*$$

where

$$s_i, u_i, v_i, \ (i = n-(k-1), \ldots, n),$$

are the k last singular values and singular vectors of the matrix $\lambda_0 I_n - \mathsf{D}$. Moreover λ_0 is an eigenvalue of Y_1 with geometric multiplicity equal to k.

Problem

The main result we obtain in this article (Theorem 4.1) generalizes this result to the case in which it is not allowed varying the whole matrix but only into a submatrix. Let G be an $n \times n$ complex matrix with less than k nonconstant invariant factors

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

partitioned into four blocks $A \in \mathbb{C}^{n_1 \times n_1}$, $B \in \mathbb{C}^{n_1 \times n_2}$, $C \in \mathbb{C}^{n_2 \times n_1}$, $D \in \mathbb{C}^{n_2 \times n_2}$.

We are going to find the distance from D to the set of matrices $Y \in \mathbb{C}^{n_2 \times n_2}$ such that the matrix

$$G_Y := \begin{pmatrix} A & B \\ C & Y \end{pmatrix}$$

is k-derogatory (in case if this set is not empty):

$$\min_{\substack{Y \in \mathbb{C}^{n_2 \times n_2} \\ G_Y \in \mathcal{M}_k^c}} \|Y - D\|. \tag{1.2}$$

Also we are going to find a matrix $Y_1 \in \mathbb{C}^{n_2 \times n_2}$ where this constrained minimum is attained.

Submatrix that lowers the rank

In order to do that we will use some results from the papers [4],[13], [18] and the book [3] which point out what are the possible ranks of all the matrices in the form

$$\mathsf{G_X} := \begin{array}{c} \begin{matrix} n_1 & n_2 \end{matrix} \\ \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{X} \end{pmatrix} \end{array} \begin{matrix} m_1 \\ m_2 \end{matrix} ,$$

by varying X in $\mathbb{C}^{m_2 \times n_2}$, and what is the nearest matrix, of this form, to the previously fixed matrix

$$\mathsf{G} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix}$$

and such that rank $\mathsf{G}_\mathsf{X} <$ rank G.

A summary of results from [13, Theorem 19, (8.1), (8.2) and (8.6)], [4, Theorem 3], [18, Theorem 2.1] and Theorem 6.3.7, page 102, of the book [3], which we need here, is the following theorem.

Theorem 1.1 *With the previous notations, let*

$$\rho := \operatorname{rank}[\mathsf{A}, \mathsf{B}] + \operatorname{rank}\begin{bmatrix}\mathsf{A}\\ \mathsf{C}\end{bmatrix} - \operatorname{rank}\mathsf{A}.$$

Then rank G_X *must satisfy*

$$\rho \leq \operatorname{rank}\mathsf{G}_\mathsf{X}$$

for all matrix $\mathsf{X} \in \mathbb{C}^{m_2 \times n_2}$.
Also it is true that there exists a matrix $\mathsf{Z} \in \mathbb{C}^{m_2 \times n_2}$ *such that*

$$\operatorname{rank}\mathsf{G}_\mathsf{Z} = \rho.$$

Now let

$$\mathsf{M} := (I - \mathsf{A}\mathsf{A}^\dagger)\mathsf{B}, \quad \mathsf{N} := \mathsf{C}(I - \mathsf{A}^\dagger\mathsf{A}),$$

then, for all $\mathsf{X} \in \mathbb{C}^{m_2 \times n_2}$,

$$\operatorname{rank}\mathsf{G}_\mathsf{X} = \rho + \operatorname{rank} S(\mathsf{X}),$$

where

$$S(\mathsf{X}) := (I - \mathsf{N}\mathsf{N}^{\dagger})(\mathsf{X} - \mathsf{C}\mathsf{A}^{\dagger}\mathsf{B})(I - \mathsf{M}^{\dagger}\mathsf{M}).$$

If r is an integer which satisfy the inequalities

$$\rho \le r < \operatorname{rank} \mathsf{G},$$

then a matrix $\mathsf{X}_0 \in \mathbb{C}^{m_2 \times n_2}$ such that

$$\|\mathsf{X}_0 - \mathsf{D}\| = \min\{\|\mathsf{X} - \mathsf{D}\| \ : \ \operatorname{rank} \mathsf{G}_{\mathsf{X}} \le r\}$$

is given by the formula

$$\mathsf{X}_0 := \mathsf{D} - U \ \operatorname{diag}\Big(0, \dots, 0, \sigma_{p+1}\big(S(\mathsf{D})\big), \sigma_{p+2}\big(S(\mathsf{D})\big), \dots, \sigma_l\big(S(\mathsf{D})\big)\Big)\, V^*, \tag{1.3}$$

in which:

(i) $p := r - \rho$,

(ii) $U \in \mathbb{C}^{m_2 \times m_2}$, $V \in \mathbb{C}^{n_2 \times n_2}$ *are the unitary matrices which appear in the singular value decomposition of the matrix* $S(\mathsf{D})$*:*

$$U^* S(\mathsf{D}) V = \operatorname{diag}\Big(\sigma_1\big(S(\mathsf{D})\big), \dots, \sigma_p\big(S(\mathsf{D})\big), \sigma_{p+1}\big(S(\mathsf{D})\big),$$

$$\dots, \sigma_l\big(S(\mathsf{D})\big)\Big) \in \mathbb{C}^{m_2 \times n_2},$$

(iii) $l := \min\{m_2, n_2\}$,

(iv) $\operatorname{diag}\Big(\sigma_1\big(S(\mathsf{D})\big), \dots, \sigma_p\big(S(\mathsf{D})\big), \sigma_{p+1}\big(S(\mathsf{D})\big), \dots, \sigma_l\big(S(\mathsf{D})\big)\Big)$ *is the* $m_2 \times n_2$ *matrix* $\Delta = (d_{ij})$, *not necessarily squared, such that*

$$d_{ij} := \begin{cases} 0 & \text{if} \quad i \neq j, \\ \sigma_i\big(S(\mathsf{D})\big) & \text{if} \quad i = j, \end{cases}$$

(v) $\operatorname{diag}\Big(0, \dots, 0, \sigma_{p+1}\big(S(\mathsf{D})\big), \sigma_{p+2}\big(S(\mathsf{D})\big), \dots, \sigma_l\big(S(\mathsf{D})\big)\Big)$ *is the matrix from (iv) with the change* $d_{ii} = 0$ *for* $i = 1, \dots, p$.

From (1.3) it results obvious that

$$\min\{\|\mathsf{X} - \mathsf{D}\| : \ \mathsf{X} \in \mathbb{C}^{m_2 \times n_2}, \ \operatorname{rank} \mathsf{G}_{\mathsf{X}} \leq r\} = \sigma_{p+1}\big(S(\mathsf{D})\big).$$

Organization

This paper is organized as follows: In Section 2 we address the question of existence of k-derogatory matrices in the shape $\left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right)$ with fixed A, B and

C and variable Y; we will see that a such matrix exists always if the size of Y is greater than or equal to k. Section 3 is devoted to an important real function h_k defined on a plane domain constituted by $\mathbb{R}^2$ minus some eigenvalues of A, if the size of D is greater than or equal to k; when this size is less than k, the definition set of the function h_k is a subset of the spectrum of A (so, it is finite). Section 4 deals with the conversion of the constrained minimization problem (1.2) in a problem of global minimization of the function h_k on its domain. Finally, in Section 5 we consider the related question of finding where are the k-derogatory eigenvalues of all matrices $\left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right)$ with Y adequately close to D; this is linked with the concept of pseudospectrum [16, 17].

2. Existence of k-derogatory matrices with constraints

In general, for a pair of matrices $(\mathsf{A}, \mathsf{B}) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times m}$ we call $i[\mathsf{A}, \mathsf{B}]$ the number of nonconstant invariant factors of the polynomial matrix $\lambda[I_n, 0] - [\mathsf{A}, \mathsf{B}]$. If $\mathsf{C} \in \mathbb{C}^{m \times n}$, we denote by $i\left[\begin{smallmatrix}\mathsf{A}\\ \mathsf{C}\end{smallmatrix}\right]$ the number of nonconstant invariant factors of the polynomial matrix $\lambda\left[\begin{smallmatrix}I_n\\ 0\end{smallmatrix}\right] - \left[\begin{smallmatrix}\mathsf{A}\\ \mathsf{C}\end{smallmatrix}\right]$. Let $\nu \in \{1, \ldots, n+m\}$. A result from [15, Theorem 6, p. 6] says that there exists a matrix $\mathsf{D} \in \mathbb{C}^{m \times m}$ such that

$$i \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} \leq \nu$$

if and only if

$$\max\left\{ i[\mathsf{A}, \mathsf{B}], i\left[\begin{smallmatrix}\mathsf{A}\\ \mathsf{C}\end{smallmatrix}\right] \right\} \leq \nu.$$

Now we turn over our problem. So, let G be an $n \times n$ complex such that $i(G) < k$,

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

partitioned into four blocks $A \in \mathbb{C}^{n_1 \times n_1}$, $B \in \mathbb{C}^{n_1 \times n_2}$, $C \in \mathbb{C}^{n_2 \times n_1}$, $D \in \mathbb{C}^{n_2 \times n_2}$. From aforementioned Silva's result [15], we have that

$$\max\left\{ i[A, B], i\left[\begin{smallmatrix}A\\ C\end{smallmatrix}\right] \right\} \leq k - 1.$$

We are going to find the distance from D to the set of matrices $Y \in \mathbb{C}^{n_2 \times n_2}$ such that the matrix

$$G_Y := \begin{pmatrix} A & B \\ C & Y \end{pmatrix}$$

is k-derogatory, in case if this set is not empty.

General considerations

For each $\lambda \in \mathbb{C}$, let $\mathcal{N}_{k,\lambda}$ be the set of all matrices $Y \in \mathbb{C}^{n_2 \times n_2}$ such that λ is an eigenvalue of G_Y with geometric multiplicity $\geq k$; with symbols,

$$\mathcal{N}_{k,\lambda} := \{Y \in \mathbb{C}^{n_2 \times n_2} :\ \operatorname{gm}(\lambda, G_Y) \geq k\}.$$

As we will see later it can occur that for some $\lambda \in \mathbb{C}$ let the set $\mathcal{N}_{k,\lambda}$ be empty. Taking this into account we define the set

$$\Omega_k := \{\lambda \in \mathbb{C} :\ \mathcal{N}_{k,\lambda} \neq \emptyset\}. \tag{2.1}$$

Or what is equivalent,

$$\Omega_k := \{\lambda \in \mathbb{C} :\ \exists Y \in \mathbb{C}^{n_2 \times n_2},\ \operatorname{gm}(\lambda, G_Y) \geq k\}.$$

In other words, Ω_k is the set of all k-derogatory eigenvalues of matrices in the shape $\left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right)$ where Y runs over $\mathbb{C}^{n_2\times n_2}$. Calling

$$\mathcal{N}_k := \{Y \in \mathbb{C}^{n_2\times n_2} : \ i(G_Y) \geq k\}, \tag{2.2}$$

we have

$$\mathcal{N}_k = \bigcup_{\lambda\in\Omega_k} \mathcal{N}_{k,\lambda}. \tag{2.3}$$

So, Ω_k is empty if and only if $\mathcal{N}_k$ is empty. Note that

$$\mathcal{N}_k = \{Y \in \mathbb{C}^{n_2\times n_2} : \ G_Y \in \mathcal{M}_k^c\}.$$

For every $\lambda \in \mathbb{C}$ we define

$$\rho_1(\lambda) := \operatorname{rank}[\lambda I_{n_1} - A, -B] + \operatorname{rank}\begin{bmatrix}\lambda I_{n_1} - A \\ -C\end{bmatrix} - \operatorname{rank}(\lambda I_{n_1} - A), \tag{2.4}$$

$$M(\lambda) := \left[I_{n_1} - (\lambda I_{n_1} - A)(\lambda I_{n_1} - A)^\dagger\right]\left(- B\right), \tag{2.5}$$

$$N(\lambda) := \left(- C\right)\left[I_{n_1} - (\lambda I_{n_1} - A)^\dagger(\lambda I_{n_1} - A)\right]. \tag{2.6}$$

Thus, the functions ρ_1, M and N depend only on $\lambda \in \mathbb{C}$. Moreover, for every $\lambda \in \mathbb{C}$ and every $Y \in \mathbb{C}^{n_2 \times n_2}$, we define the matrix

$$\begin{aligned} S_1(\lambda, Y) &:= \\ &\big(I_{n_2} - N(\lambda)N(\lambda)^\dagger\big)\big(\lambda I_{n_2} - Y - C(\lambda I_{n_1} - A)^\dagger B\big) \\ &\times \big(I_{n_2} - M(\lambda)^\dagger M(\lambda)\big). \end{aligned} \tag{2.7}$$

by Theorem 1.1, it follows

$$\begin{aligned} \operatorname{rank}(\lambda I_n - G_Y) &= \operatorname{rank}\left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & \lambda I_{n_2} - Y \end{array}\right) \\ &= \rho_1(\lambda) + \operatorname{rank} S_1(\lambda, Y). \end{aligned} \tag{2.8}$$

First of all, we deduce a lower bound of the function ρ_1 due to the hypothesis $i(G) < k$.

Proposition 2.1 *With the preceding notations, for all $\lambda \in \mathbb{C}$,*

$$n_1 - k + 1 \leq \rho_1(\lambda).$$

Proof: By (2.8) we have

$$\operatorname{rank}\left(\begin{array}{c|c}\lambda I_{n_1}-A & -B \\ \hline -C & \lambda I_{n_2}-D\end{array}\right)=\rho_1(\lambda)+\operatorname{rank} S_1(\lambda, D). \tag{2.9}$$

As $i(G)<k$, for all $\lambda \in \mathbb{C}$,

$$n-k<\operatorname{rank}\left(\begin{array}{c|c}\lambda I_{n_1}-A & -B \\ \hline -C & \lambda I_{n_2}-D\end{array}\right)=\rho_1(\lambda)+\operatorname{rank} S_1(\lambda, D), \tag{2.10}$$

So, by (2.9) and (2.10),

$$n_1+n_2-k<\rho_1(\lambda)+\operatorname{rank} S_1(\lambda, D);$$

since $\operatorname{rank} S_1(\lambda, D) \leq n_2$, we have

$$n_1+n_2-k<\rho_1(\lambda)+n_2.$$

What implies $n_1-k<\rho_1(\lambda)$ or, equivalently, $n_1-k+1 \leq \rho_1(\lambda)$. □

Proposition 2.2

$$\Omega_k=\{\lambda \in \mathbb{C}: \ \rho_1(\lambda) \leq n-k\}.$$

Proof: If $\lambda \in \Omega_k$, then there exists $Y \in \mathbb{C}^{n_2 \times n_2}$ such that $\mathrm{gm}(\lambda, G_Y) \geq k$; what is equivalent to $\mathrm{rank}(\lambda I_n - G_Y) \leq n - k$. Hence by (2.8),

$$\rho_1(\lambda) + \mathrm{rank}\, S_1(\lambda, Y) \leq n - k.$$

Therefore $\rho_1(\lambda) \leq n - k$.

Conversely, if $\lambda \in \mathbb{C}$ is such that $\rho_1(\lambda) \leq n - k$, then taking

$$Y_\lambda := \lambda I_{n_2} - C(\lambda I_{n_1} - A)^\dagger B,$$

it follows by (2.7) that $S_1(\lambda, Y_\lambda) = 0$. Thus,

$$\mathrm{rank}(\lambda I_{n_2} - G_{Y_\lambda}) = \rho_1(\lambda);$$

so,

$$\mathrm{rank}(\lambda I_{n_2} - G_{Y_\lambda}) \leq n - k.$$

Consequently, $\mathcal{N}_{k,\lambda} \neq \emptyset$. Hence $\lambda \in \Omega_k$. □

If k is greater than n_2, then the set Ω_k is "small" as we are going to see next.

Proposition 2.3 *If $n_2 < k$ and $\lambda \notin \Lambda(A)$, then $\mathcal{N}_{k,\lambda} = \emptyset$.*

Proof: Given that every eigenvalue of (A, B), resp. of (C, A), is an eigenvalue of A, from (2.4) for all $\lambda \notin \Lambda(A)$ we have $\rho_1(\lambda) = n_1$. And if there exists a matrix $Y \in \mathcal{N}_{k,\lambda}$ it follows from definition of $\mathcal{N}_{k,\lambda}$ and (2.8) that

$$\rho_1(\lambda) + \operatorname{rank} S_1(\lambda, Y) \leq n - k;$$

hence $n_1 \leq n_1 + n_2 - k$. This implies $0 \leq n_2 - k$, which contradicts $n_2 - k < 0$. □

Thus, from this proposition and the definition (2.1) of the set Ω_k we can derive the following result.

Proposition 2.4 *If $n_2 < k$, then*

$$\Omega_k \subset \Lambda(A).$$

So, when $n_2 < k$, what eigenvalues of A belong to Ω_k? we will answer this question later on. Before, let us establish a sufficient condition for Ω_k to be empty.

Proposition 2.5 *Suppose that $n_2 < k$ and for all $\alpha \in \Lambda(A)$ we have*

$$\mathrm{gm}(\alpha, A) < k - n_2. \tag{2.11}$$

Then

$$\Omega_k = \emptyset.$$

Proof: If for all $\alpha \in \Lambda(A)$, $\quad \mathrm{gm}(\alpha, A) < k - n_2$, then

$$\mathrm{rank}(\alpha I_{n_1} - A) > n_1 - (k - n_2) = n_1 + n_2 - k = n - k;$$

therefore, $\rho_1(\alpha) > n - k$. Hence, by Proposition 2.2 $\alpha \notin \Omega_k$ and, as $\Omega_k \subset \Lambda(A)$ from Proposition 2.4, we deduce

$$\Omega_k = \emptyset.$$

□

Proposition 2.5 admits the next equivalent statements.

Proposition 2.6 *Suppose that $n_2 < k$ and for all $\alpha \in \Lambda(A)$ we have*

$$\mathrm{gm}(\alpha, A) < k - n_2.$$

Then, it does not exist any matrix $Y \in \mathbb{C}^{n_2 \times n_2}$ such that

$$i \begin{pmatrix} A & B \\ C & Y \end{pmatrix} \geq k;$$

i.e. the set $\mathcal{N}_k$ is empty.

Examples with $n_2 < k$

We are going to consider two examples that show us the set Ω_k can be empty.

Example 2.1 Let

$$G := \begin{pmatrix} A & B \\ C & D \end{pmatrix} := \left(\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ \hline 1 & 0 & -1 & 0 & 2 \\ -1 & 0 & 1 & 1 & 0 \end{array}\right).$$

Here $n_2 = 2, n = 5$. Let $k := 4$. Since $i(G) = 1$, we have $i(G) < k$; by the other hand, $\Lambda(A) = \{0, 1\}$. We see that $\operatorname{gm}(0, A) = 1, \operatorname{gm}(1, A) = 1$; so,

$1 < 2 = 4 - 2 = k - n_2$, but $\rho_1(0) = 4$ and $\rho_1(1) = 4$; hence

$$\rho_1(0) \not\leq 1 = 5 - 4 = n - k \text{ so, Proposition 2.2 implies } 0 \notin \Omega_4$$
$$\rho_1(1) \not\leq 1 = 5 - 4 = n - k \text{ so, Proposition 2.2 implies } 1 \notin \Omega_4$$

Hence, by Proposition 2.4,

$$\Omega_4 = \emptyset.$$

As $k - n_2 = 4 - 2 = 2$, this result can also be deduced directly from Proposition 2.5 without the need to compute $\rho_1(0), \rho_1(1)$.

Notwithstanding it can occur that $\Omega_k = \emptyset$ though for some $\alpha \in \Lambda(A)$ we have $\operatorname{gm}(\alpha, A) \geq k - n_2$, as we can see in the next example.

Example 2.2 Let

$$G := \begin{pmatrix} A & B \\ C & D \end{pmatrix} := \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ \hline 1 & 0 & -1 & 0 & 2 \\ -1 & 0 & 1 & 1 & 0 \end{array}\right).$$

Here we again take $k := 4$; given that $i(G) = 2$ it follows $i(G) < k$. Now $\Lambda(A) = \{0\}$, and

$$\mathrm{gm}(0, A) = 2 \not< 2 = k - n_2;$$

but

$$\rho_1(0) = 3 \not\leq 1 = 5 - 4 = n - k.$$

Thus, by Propositions 2.4 and 2.2, $0 \notin \Omega_4$.

Therefore, condition (2.11) of Proposition 2.5 is sufficient for $\Omega_k = \emptyset$, but it is not a necessary condition. However, the condition $n_2 < k$ is necessary for $\Omega_k = \emptyset$, as we will see in Proposition 2.7.

Existence of k-derogatory matrices when $n_2 \geq k$

In the previous examples, $n_2 < k$. Let us see that when $n_2 \geq k$, the situation changes. The following proposition give us a sufficient condition so that the set in (2.2) is not empty.

Proposition 2.7 *Let n_1, n_2 be positive integers, let $A \in \mathbb{C}^{n_1 \times n_1}$, $B \in \mathbb{C}^{n_1 \times n_2}$, $C \in \mathbb{C}^{n_2 \times n_1}$ and let $n := n_1 + n_2$. Let k, $2 \leq k \leq n$, be an integer.*

If $n_2 \geq k$ then there exist matrices $Y \in \mathbb{C}^{n_2 \times n_2}$ such that the matrix

$$G_Y := \begin{pmatrix} A & B \\ C & Y \end{pmatrix} \in \mathbb{C}^{n \times n}$$

is k-derogatory.

Proof: Let λ_0 be a complex number which is not an eigenvalue of A. Let

$$Y_0 := \lambda_0 I_{n_2} - C(\lambda_0 I_{n_1} - A)^{-1} B \tag{2.12}$$

By virtue of Theorem 1.1,

$$\begin{aligned}\operatorname{rank}(\lambda_0 I_n - G_{Y_0}) &= \operatorname{rank}\left(\begin{array}{c|c}\lambda_0 I_{n_1} - A & -B \\ \hline -C & \lambda_0 I_{n_2} - Y_0\end{array}\right) \\ &= \operatorname{rank}[\lambda_0 I_{n_1} - A, -B] + \operatorname{rank}\begin{bmatrix}\lambda_0 I_{n_1} - A \\ -C\end{bmatrix} - \operatorname{rank}(\lambda_0 I_{n_1} - A) \\ &\quad + \operatorname{rank}(\lambda_0 I_{n_2} - Y_0 - C(\lambda_0 I_{n_1} - A)^{-1}B) \\ &= n_1 + n_1 - n_1 + \operatorname{rank}(\lambda_0 I_{n_2} - Y_0 - C(\lambda_0 I_{n_1} - A)^{-1}B) \\ &= n_1 + \operatorname{rank}(\lambda_0 I_{n_2} - \lambda_0 I_{n_2} + C(\lambda_0 I_{n_1} - A)^{-1}B - \\ &\quad C(\lambda_0 I_{n_1} - A)^{-1}B) \\ &= n_1\end{aligned}$$

As $k \leq n_2$, we have $n_1 \leq n_1 + n_2 - k = n - k$. Therefore λ_0 is a k-derogatory eigenvalue of G_{Y_0} and this matrix is k-derogatory. □

Remark 2.1 Note that this proposition proves even more: For each $\lambda \in \mathbb{C} \setminus \Lambda(A)$ there exists a matrix $Y_\lambda \in \mathbb{C}^{n_2 \times n_2}$ such that λ is a k-derogatory eigenvalue of G_{Y_λ}.

Existence of k-derogatory matrices when $n_2 < k$

After Proposition 2.4 we write the following question: When $n_2 < k$, what eigenvalues of A belong to Ω_k? An answer is $\lambda_0 \in \Lambda(A)$ belongs to Ω_k if and only if $\rho_1(\lambda_0) \leq n-k$, as it can be seen from the final part of the proof of the next result ("if") and Proposition 2.2 ("only if").

Proposition 2.8 *If $n_2 < k$, then there exists a $Y \in \mathbb{C}^{n_2 \times n_2}$ such that $i(G_Y) \geq k$ if and only if there exists a $\lambda_0 \in \Lambda(A)$ such that $\rho_1(\lambda_0) \leq n-k$.*
PROOF: If there exists a $Y \in \mathbb{C}^{n_2 \times n_2}$ such that $i(G_Y) \geq k$, then G_Y has a k-derogatory eigenvalue α; i.e. $\mathrm{gm}(\alpha, G_Y) \geq k$. Hence

$$\operatorname{rank}(\alpha I_n - G_Y) \leq n - k. \tag{2.13}$$

By (2.8)

$$\operatorname{rank}(\alpha I_n - G_Y) = \rho_1(\alpha) + \operatorname{rank} S_1(\alpha, Y). \tag{2.14}$$

Therefore, $\alpha \in \Lambda(A)$; otherwise, given that $0 \leq \operatorname{rank} S_1(\alpha, Y)$, from (2.13) and (2.14) we should have $k \leq n_2$; what is absurd. Moreover, (2.13) and (2.14) imply $\rho_1(\alpha) \leq n-k$.

Conversely, if there exists a $\lambda_0 \in \Lambda(A)$ such that $\rho_1(\lambda_0) \leq n-k$ take

$$Y := \lambda_0 I_{n_2} - C(\lambda_0 I_{n_1} - A)^{\dagger} B;$$

by (2.7), this implies $S_1(\lambda_0, Y) = 0$. Hence, by (2.8),

$$\operatorname{rank}(\lambda_0 I_n - G_Y) = \rho_1(\lambda_0);$$

so, $\operatorname{gm}(\lambda_0, G_Y) \geq k$. □

Paraphrasing this statement in analogous terms to those of [15], we have:

Proposition 2.9 *Let $A \in \mathbb{C}^{n_1 \times n_1}, B \in \mathbb{C}^{n_1 \times n_2}, C \in \mathbb{C}^{n_2 \times n_1}$. Let k be an integer, $n_2 < k \leq n$. Then there exists a $Y \in \mathbb{C}^{n_2 \times n_2}$ such that*

$$i \begin{pmatrix} A & B \\ C & Y \end{pmatrix} \geq k$$

if and only if there exists a $\lambda_0 \in \Lambda(A)$ such that

$$\operatorname{rank}[\lambda_0 I_{n_1} - A, -B] + \operatorname{rank} \begin{bmatrix} \lambda_0 I_{n_1} - A \\ -C \end{bmatrix} - \operatorname{rank}(\lambda_0 I_{n_1} - A) \leq n - k.$$

Remark 2.2 The reference to the Moore-Penrose inverse in this statement and many of the results of this paper referring to ranks can be weakened. According to Theorem 6.3.7, page 102, of the book [3] and Theorem 19 from [13], we can put A^- instead of $\mathsf{A}^\dagger$, where A^- is any (1)-inverse of the matrix $\mathsf{A} \in \mathbb{C}^{m\times n}$; that is to say, A^- is any solution of the equation

$$\mathsf{AXA} = \mathsf{A}.$$

Scalar matrices

Consider now the case $k = n$. Given the matrices $A \in \mathbb{C}^{n_1\times n_1}$, $B \in \mathbb{C}^{n_1\times n_2}$, $C \in \mathbb{C}^{n_2\times n_1}$, $D \in \mathbb{C}^{n_2\times n_2}$ such that

$$i\begin{pmatrix} A & B \\ C & D \end{pmatrix} < n,$$

what conditions must satisfy A, B and C for there exists a matrix $Y \in \mathbb{C}^{n_2\times n_2}$ such that

$$i\begin{pmatrix} A & B \\ C & Y \end{pmatrix} \geq n?$$

This question is equivalent to ask for conditions to

$$i\begin{pmatrix} A & B \\ C & Y \end{pmatrix} = n$$

or, what is the same, that the matrix $\left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right)$ se a scalar matrix. Recall that a scalar matrix is a matrix in the shape αI_n with an $\alpha \in \mathbb{C}$. If there exists $Y \in \mathbb{C}^{n_2 \times n_2}$ such that $\left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right)$ is a scalar matrix, then necessarily there is an $\alpha \in \mathbb{C}$ such that $\left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right) = \alpha I_n$; hence it follows

$$\begin{aligned} A = \alpha I_{n_1}, \quad & B = 0 \\ C = 0, \qquad & Y = \alpha I_{n_2}. \end{aligned}$$

Therefore, for the existence of Y such that $i\left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right) \geq n$ are necessary conditions that A, B and C be in the shape

$$\begin{aligned} A = \alpha I_{n_1}, \text{ for some } \alpha \in \mathbb{C}, \\ B = 0, \\ C = 0. \end{aligned} \tag{2.15}$$

Let see that these conditions (2.15) are also sufficient. In fact, under them there exists $Y := \alpha I_{n_2}$ such that

$$\begin{pmatrix} \alpha I_{n_1} & 0 \\ 0 & \alpha I_{n_2} \end{pmatrix}$$

is a scalar matrix.

Reminding that

$$\Omega_k = \{\lambda \in \mathbb{C} : \rho_1(\lambda) \leq n - k\}$$

by Proposition 2.2, and assigning the value n to k, one has

$$\Omega_n = \{\lambda \in \mathbb{C} : \rho_1(\lambda) \leq n - n\};$$

so, $\lambda \in \Omega_n \Leftrightarrow \rho_1(\lambda) = 0$. Now then, $\rho_1(\lambda) = 0$ is equivalent to

$$\operatorname{rank}(\lambda I_{n_1} - A) = 0, \quad \operatorname{rank}(-B) = 0, \quad \operatorname{rank}(-C) = 0;$$

and these conditions are equivalent to

$$A = \lambda I_{n_1}, \quad B = 0, \quad C = 0.$$

Thus, the set Ω_n has only an element: the one $\lambda \in \mathbb{C}$ such that $A = \lambda I_{n_1}$. Therefore,

$$\Omega_n = \{\alpha\}. \tag{2.16}$$

3. The function of two real variables to be minimized

Let k be an integer, $2 \leq k \leq n$. Let

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{(n_1+n_2)\times(n_1+n_2)},$$

be an $n \times n$, four block partitioned matrix such that $i(G) < k$. In order to use the Theorem 1.1, for each $\lambda \in \Omega_k$ we define

$$p_k(\lambda) := n - k - \rho_1(\lambda). \tag{3.1}$$

Proposition 3.1 *For all* $\lambda \in \Omega_k$, $\quad 0 \leq p_k(\lambda) \leq n_2 - 1.$

Proof: From the definition of $p_k(\lambda)$ and Proposition 2.2 for all $\lambda \in \Omega_k$ we have $0 \leq p_k(\lambda)$. By Proposition 2.1,

$$\begin{aligned} p_k(\lambda) &= n_1 + n_2 - k - \rho_1(\lambda) \leq n_1 + n_2 - k - (n_1 - k + 1) \\ &= n_1 + n_2 - k - n_1 + k - 1 = n_2 - 1. \end{aligned}$$

□

Let

$$\begin{array}{rccl} h_k: & \Omega_k & \to & \mathbb{R} \\ & \lambda & \mapsto & \sigma_{p_k(\lambda)+1}\big(S_1(\lambda, D)\big) \end{array} \qquad (3.2)$$

be the function that associates to each complex number $\lambda \in \Omega_k$ the $(p_k(\lambda)+1)$th singular value of the $n_2 \times n_2$ matrix $S_1(\lambda, D)$. The definition of this matrix can be seen in (2.7) changing Y by D.

Let us now assume that $n_2 \geq k$, which is the most interesting case. Theorem 3.3 summarizes some properties of the function h_k. Before of giving its statement, we need some previous results.

Lemma 3.2 *Let M_1, M_2, M_3 be $n \times n$ complex matrices. Let k be an integer, $2 \leq k \leq n$. Then the following inequalities concerning their singular values are true:*

(i) $\sigma_n(M_1)\, \sigma_{n-k+1}(M_2)\, \sigma_n(M_3) \leq \sigma_{n-k+1}(M_1 M_2 M_3)$,

(ii) $\sigma_{n-k+1}(M_1 M_2 M_3) \leq \|M_1\|\, \|M_3\|\, \sigma_{n-k+1}(M_2)$.

Proof: The inequalities in each line follow from two applications of Theorem 1, p. 44, of [14]. □

Now let

$$F(\lambda) := \lambda I_{n_2} - D - C(\lambda I_{n_1} - A)^{-1}B;$$

here, $\lambda I_{n_2} - D$ is a polynomial matrix in the variable λ and $C(\lambda I_{n_1} - A)^{-1}B$ is a strictly proper rational matrix function because

$$\lim_{|\lambda|\to\infty} C(\lambda I_{n_1} - A)^{-1}B = 0.$$

Moreover, for each

$$\lambda \in \mathbb{C} \setminus \left(\Lambda(A) \cup \Lambda\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)\right),$$

we have

$$n = \operatorname{rank}\left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & \lambda I_{n_2} - D \end{array}\right) =$$

$$\operatorname{rank}(\lambda I_{n_1} - A) + \operatorname{rank} F(\lambda) = n_1 + \operatorname{rank} F(\lambda),$$

in virtue of formula (7), p. 46, of [12] on the Schur complement of $\lambda I_{n_1} - A$ in

$$\left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & \lambda I_{n_2} - D \end{array}\right).$$

Hence,

$$\operatorname{rank} F(\lambda) = n_2$$

and so $\det F(\lambda) \not\equiv 0$. Therefore, we can consider the *local Smith form* of the rational matrix function $F(\lambda)$ at λ_0, the complex number λ_0 being an eigenvalue of A:

$$F(\lambda) = E_1(\lambda)\ \operatorname{diag}[(\lambda-\lambda_0)^{\nu_1}, \ldots, (\lambda-\lambda_0)^{\nu_{n_2}}]\ E_2(\lambda), \qquad (3.3)$$

where $E_1(\lambda)$ and $E_2(\lambda)$ are rational matrix functions that are defined and invertible at λ_0, and $\nu_1, \ldots, \nu_{n_2}$ are integers; these integers are uniquely determined by $F(\lambda)$ and λ_0 up to permutation and do not depend on the particular choice of the local Smith form (3.3); they are called the *partial multiplicities* of $F(\lambda)$ at λ_0. See Section 7.2, p. 218–219, of [6].

In virtue of Theorem 1.13.2 (3), p. 25, of the book [10], the poles of $F(\lambda)$ belong to $\Lambda(A)$ even if the realization $C(\lambda I_{n_1} - A)^{-1}B$ is not minimal. But it may occur that some eigenvalues of A are not poles of $F(\lambda)$.

Theorem 3.3 *With the previous notations, let us assume $n_2 \geq k$. Let $h_k : \Omega_k \to \mathbb{R}$ be the function we have defined in (3.2). Then*

(i) *the function h_k is continuous on $\Omega_k \setminus \Lambda(A)$,*

(ii) *if* $\lambda_0 \in \Lambda(A)$ *and the number of negative partial multiplicities of* $F(\lambda)$ *at* λ_0 *is greater than or equal to* $n_2 - k + 1$*, then*

$$\lim_{\lambda \to \lambda_0} h_k(\lambda) = \infty,$$

(iii) *if* $\lambda_0 \in \Lambda(A)$ *and the number of negative partial multiplicities of* $F(\lambda)$ *at* λ_0 *is less than* $n_2 - k + 1$*, then there exists the limit*

$$\lim_{\lambda \to \lambda_0} h_k(\lambda),$$

(iv)

$$\lim_{|\lambda| \to \infty} h_k(\lambda) = \infty.$$

PROOF:

(i) If $\lambda \in \Omega_k \setminus \Lambda(A)$, then

$$(\lambda I_{n_1} - A)^{\dagger} = (\lambda I_{n_1} - A)^{-1}$$

and therefore

$$M(\lambda) = 0, \quad N(\lambda) = 0$$

and from (3.1) it follows

$$p_k(\lambda) = n_1 + n_2 - k - n_1 = n_2 - k; \tag{3.4}$$

so that

$$h_k(\lambda) = \sigma_{n_2-k+1}\big(\lambda I_{n_2} - D - C(\lambda I_{n_1} - A)^{-1}B\big). \tag{3.5}$$

By virtue of the continuity of the function

$$\lambda \mapsto (\lambda I_{n_1} - A)^{-1}$$

on $\mathbb{C} \setminus \Lambda(A)$ and because of being the singular values of a matrix continuous functions of it, it follows that the function

$$\lambda \mapsto \sigma_{n_2-k+1}\big(\lambda I_{n_2} - D - C(\lambda I_{n_1} - A)^{-1}B\big)$$

is continuous at each point $\lambda \in \Omega_k \setminus \Lambda(A)$.

(ii) Call $\Delta(\lambda)$ the diagonal matrix

$$\operatorname{diag}[(\lambda - \lambda_0)^{\nu_1}, \ldots, (\lambda - \lambda_0)^{\nu_{n_2}}]$$

that appears in (3.3). Applying inequality (i) of Lemma 3.2 to product (3.3) we have

$$\sigma_{n_2}\big(E_1(\lambda)\big)\ \sigma_{n_2-k+1}\big(\Delta(\lambda)\big)\ \sigma_{n_2}\big(E_2(\lambda)\big) \leq \sigma_{n_2-k+1}\big(F(\lambda)\big). \qquad (3.6)$$

It is easy to see that the singular values of $\Delta(\lambda)$ are

$$|\lambda-\lambda_0|^{\nu_1}, \ldots, |\lambda-\lambda_0|^{\nu_{n_2}},$$

(not necessarily ordered from largest to smallest). By the hypothesis on the negative partial multiplicities of $F(\lambda)$ at λ_0, we have that the (n_2-k+1)th singular value of $\Delta(\lambda)$ (when ordered in nonincreasing order) is in the shape

$$\frac{1}{|\lambda-\lambda_0|^p},$$

with a positive integer p (the number p does not depend on λ!) for all λ *sufficiently* closed to λ_0 and different from it. Hence,

$$\lim_{\lambda\to\lambda_0} \sigma_{n_2-k+1}\big(\Delta(\lambda)\big) = \lim_{\lambda\to\lambda_0} \frac{1}{|\lambda-\lambda_0|^p} = \infty. \qquad (3.7)$$

As $E_1(\lambda_0)$ and $E_2(\lambda_0)$ are invertible it follows

$$\lim_{\lambda\to\lambda_0} \sigma_{n_2}\big(E_1(\lambda)\big) = \sigma_{n_2}\big(E_1(\lambda_0)\big) > 0,$$

$$\lim_{\lambda\to\lambda_0} \sigma_{n_2}\big(E_2(\lambda)\big) = \sigma_{n_2}\big(E_2(\lambda_0)\big) > 0.$$

Therefore, by (3.7) we have

$$\lim_{\lambda\to\lambda_0} \sigma_{n_2}\big(E_1(\lambda)\big)\, \sigma_{n_2-k+1}\big(\Delta(\lambda)\big)\, \sigma_{n_2}\big(E_2(\lambda)\big) = \infty;$$

from here and (3.6) it follows

$$\lim_{\lambda\to\lambda_0} \sigma_{n_2-k+1}\big(F(\lambda)\big) = \infty.$$

(iii) Let q be the number of negative partial multiplicities of $F(\lambda)$ at λ_0. So, $q < n_2 - k + 1$. Permuting the elements of the diagonal of $\Delta(\lambda)$, if necessary, we can suppose that

$$\nu_1 < 0, \dots, \nu_q < 0, \nu_{q+1} \geq 0, \dots, \nu_{n_2} \geq 0.$$

Then the singular values of $\Delta(\lambda)$ are

$$\frac{1}{|\lambda-\lambda_0|^{-\nu_1}}, \dots, \frac{1}{|\lambda-\lambda_0|^{-\nu_q}}, |\lambda-\lambda_0|^{\nu_{q+1}}, \dots, |\lambda-\lambda_0|^{\nu_{n_2}}. \qquad (3.8)$$

In the case of λ is sufficiently close to λ_0, the numbers

$$\frac{1}{|\lambda-\lambda_0|^{-\nu_1}},\ldots,\frac{1}{|\lambda-\lambda_0|^{-\nu_q}},$$

are the q greatest numbers in the list (3.8); thus,

$$\sigma_{n_2-k+1}\big(\Delta(\lambda)\big)=|\lambda-\lambda_0|^{\ell} \tag{3.9}$$

with ℓ an integer ≥ 0 (the number ℓ does not depend on λ!).

Taking into account (3.3), (3.9) and inequality (ii) in Lemma 3.2,

$$\begin{aligned}\sigma_{n_2-k+1}\big(F(\lambda)\big) &\leq \|E_1(\lambda)\|\,\|E_2(\lambda)\|\,\sigma_{n_2-k+1}\big(\Delta(\lambda)\big)\\ &= \|E_1(\lambda)\|\,\|E_2(\lambda)\|\,|\lambda-\lambda_0|^{\ell}.\end{aligned} \tag{3.10}$$

Given that $E_1(\lambda_0)$ and $E_2(\lambda_0)$ are invertible,

$$\|E_1(\lambda_0)\|>0,\|E_2(\lambda_0)\|>0;$$

then by (3.10) there exist a real number $M>0$ and a deleted neighbourhood $\mathcal{N}$ of λ_0 such that for all $\lambda\in\mathcal{N}$, we have

$$\sigma_{n_2-k+1}\big(F(\lambda)\big)\leq M.$$

From this upper bound and due to the fact that $\sigma_{n_2-k+1}\big(F(\lambda)\big)$ is an algebraic function, it follows that there exists the limit

$$\lim_{\lambda\to\lambda_0} \sigma_{n_2-k+1}\big(F(\lambda)\big).$$

(iv) For all $\lambda \in \mathbb{C} \setminus \Lambda(A)$, by [9, p.178, Theorem 3.3.16 (c)] we have

$$\begin{aligned}\Big|\sigma_{n_2-k+1}\big(\lambda I_{n_2} - D - C(\lambda I_{n_1} - A)^{-1}B\big) - \sigma_{n_2-k+1}\big(\lambda I_{n_2} - D\big)\Big| \\ \leq \| - C(\lambda I_{n_1} - A)^{-1}B\|. \end{aligned} \tag{3.11}$$

As $(\lambda I_{n_1} - A)^{-1}$ is a matrix of strictly proper rational functions in λ, we have

$$\| - C(\lambda I_{n_1} - A)^{-1}B\| \to 0 \tag{3.12}$$

when $|\lambda| \to \infty$. Given that $\sigma_{n_2-k+1}(\lambda I_{n_2} - D) \to \infty$ when $|\lambda| \to \infty$ [7, proof of Theorem 4.1], it follows from (3.11) and (3.12) that

$$\lim_{|\lambda|\to\infty} \sigma_{n_2-k+1}\big(\lambda I_{n_2} - D - C(\lambda I_{n_1} - A)^{-1}B\big) = \infty.$$

□

Remark 3.1 From this theorem it follows that there exists the minimum

$$\min_{\lambda\in\Omega_k} h_k(\lambda). \tag{3.13}$$

If $n_2 < k$, by Proposition 2.4 the set Ω_k is finite; from which the minimum (3.13) exists for whatever value of k.

In relation with point (iii) of the proof of Theorem 3.3, when λ_0 is not a pole of $F(\lambda)$, i.e. there is no negative partial multiplicity of λ_0, we can say more.

Theorem 3.4 *With the preceding notations, if λ_0 is not a pole of $F(\lambda)$, then*

$$\lim_{\lambda\to\lambda_0} \sigma_{n_2-k+1}\big(F(\lambda)\big) = \sigma_{n_2-k+1}(\lambda_0 I_{n_2} - D - C S_{\lambda_0} B);$$

where

$$S_{\lambda_0} = \sum_{\alpha\in\Lambda(A)\setminus\{\lambda_0\}} \left[\frac{P_\alpha}{\lambda_0-\alpha} + \frac{D_\alpha}{(\lambda_0-\alpha)^2} + \cdots + \frac{D_\alpha^{\nu(\alpha)-1}}{(\lambda_0-\alpha)^{\nu(\alpha)}}\right]$$

For each $\alpha \in \Lambda(A)$, the matrices P_α, D_α and the number $\nu(\alpha)$ are the Riesz eigenprojection, the eigennilpotent and the index, belonging to the eigenvalue α, respectively.

Proof: The Laurent expansion of the resolvent of A in a neighbourhood of λ_0 is

$$(\lambda I_{n_1} - A)^{-1} = \frac{P_{\lambda_0}}{\lambda - \lambda_0} + \sum_{j=2}^{\nu(\lambda_0)} \frac{D_{\lambda_0}^{j-1}}{(\lambda - \lambda_0)^j} + \sum_{n=0}^{\infty} (-1)^n S_{\lambda_0}^{n+1} \cdot (\lambda - \lambda_0)^n. \quad (3.14)$$

where

$$P_{\lambda_0} := \frac{1}{2\pi i} \oint_\Gamma (\lambda I_{n_1} - A)^{-1} \, \mathrm{d}\lambda,$$

Γ being a suitable sufficiently small positively oriented circle centred at λ_0. The matrix P_{λ_0} is the Riesz projector or eigenprojection associated to λ_0. The matrix D_{λ_0} is the eigennilpotent matrix

$$D_{\lambda_0} := (A - \lambda_0 I_{n_1}) P_{\lambda_0},$$

associated to λ_0. See [1, p. 74, p.66–67][11, p. 41–42]. From (3.14), we deduce

$$F(\lambda) = \lambda I_{n_2} - D - \frac{CP_{\lambda_0}B}{\lambda - \lambda_0} - \sum_{j=2}^{\nu(\lambda_0)} \frac{CD_{\lambda_0}^{j-1}B}{(\lambda - \lambda_0)^j}$$
$$- \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_0)^n CS_{\lambda_0}^{n+1} B \qquad (3.15)$$

with

$$S_{\lambda_0} := \frac{1}{2\pi i} \oint_\Gamma \frac{1}{\lambda - \lambda_0} (\lambda I_{n_1} - A)^{-1} \, \mathrm{d}\lambda.$$

As λ_0 is not a pole of $F(\lambda)$, we have that all coefficients of negative powers of $\lambda - \lambda_0$ in (3.15) are zero. So,

$$F(\lambda) = \lambda I_{n_2} - D - \sum_{n=0}^{\infty} (-1)^n CS_{\lambda_0}^{n+1} B \cdot (\lambda - \lambda_0)^n; \qquad (3.16)$$

from it follows

$$\lim_{\lambda \to \lambda_0} \sigma_{n_2-k+1}\big(F(\lambda)\big) = \sigma_{n_2-k+1}(\lambda_0 I_{n_2} - D - CS_{\lambda_0}B).$$

From [12, p. 315] (where it puts Z_{k0} instead of P_{λ_0}), we have

$$P_{\lambda_0} = \varphi_{\lambda_0}(A) \text{ with } \varphi_{\lambda_0}(\lambda) := \frac{\prod_{j=1}^{s-1}(\lambda - \lambda_j)}{\prod_{j=1}^{s-1}(\lambda_0 - \lambda_j)},$$

with

$$\{\lambda_1, \ldots, \lambda_{s-1}\} := \Lambda(A) \smallsetminus \{\lambda_0\}.$$

The formula

$$S_{\lambda_0} = \sum_{\alpha \in \Lambda(A) \smallsetminus \{\lambda_0\}} \left[\frac{P_\alpha}{\lambda_0 - \alpha} + \frac{D_\alpha}{(\lambda_0 - \alpha)^2} + \cdots + \frac{D_\alpha^{\nu(\alpha)-1}}{(\lambda_0 - \alpha)^{\nu(\alpha)}} \right]$$

for S_{λ_0} it can be seen in [11, p. 42, (5.32)] (Kato defined the resolvent of A as $(A - \lambda I_{n_1})^{-1}$; hence the minus sign which appears in its formula (5.32)). □

Remark 3.2 The index, $\nu(\alpha)$, of each eigenvalue α of A satisfies that

$$D_\alpha \neq 0, \ldots, D_\alpha^{\nu(\alpha)-1} \neq 0 \quad \text{and} \quad D_\alpha^{\nu(\alpha)} = 0.$$

4. Optimal submatrix that increases the geometric multiplicity

Let n_1, n_2 be positive integers and $n := n_1 + n_2$. Let k be an integer, $2 \le k \le n$. Let $A \in \mathbb{C}^{n_1 \times n_1}$, $B \in \mathbb{C}^{n_1 \times n_2}$, $C \in \mathbb{C}^{n_2 \times n_1}$ and $D \in \mathbb{C}^{n_2 \times n_2}$ be matrices such that $i(G) < k$, where

$$G := \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

For each $Y \in \mathbb{C}^{n_2 \times n_2}$ let $G_Y := \left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right)$. In this section we give a solution to the problem of finding the minimum of the set

$$\{\|Y - D\| : \; Y \in \mathbb{C}^{n_2 \times n_2}, \quad i(G_Y) \ge k\} \tag{4.1}$$

by means of the following theorem.

Theorem 4.1 *Using the preceding notation, let* $A \in \mathbb{C}^{n_1 \times n_1}$, $B \in \mathbb{C}^{n_1 \times n_2}$, $C \in \mathbb{C}^{n_2 \times n_1}$, $D \in \mathbb{C}^{n_2 \times n_2}$ *be matrices such that the* $n \times n$ *matrix*

$$G := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

satisfies $i(G) < k$. *Then*

$$\min_{\substack{Y \in \mathbb{C}^{n_2 \times n_2} \\ i(G_Y) \geq k}} \|Y - D\| = \min_{\lambda \in \Omega_k} h_k(\lambda). \tag{4.2}$$

Moreover, if λ_0 *is a complex number where the function* $h_k : \Omega_k \to \mathbb{R}$ *attains its minimum value, then a matrix* Y_1 *which minimizes the left-hand side of (4.2) is given by*

$$Y_1 := D + U \operatorname{diag}(0, \ldots, 0, \tau_{p_k(\lambda_0)+1}, \ldots, \tau_{n_2})V^*, \tag{4.3}$$

where $U, V \in \mathbb{C}^{n_2 \times n_2}$ *are the unitary matrices which appear into the singular value decomposition of the matrix* $S_1(\lambda_0, D)$*:*

$$U^* S_1(\lambda_0, D) V = \operatorname{diag}(\tau_1, \ldots, \tau_{p_k(\lambda_0)}, \tau_{p_k(\lambda_0)+1}, \ldots, \tau_{n_2}). \tag{4.4}$$

And λ_0 *is also a* k*-derogatory eigenvalue of the matrix* G_{Y_1}*; in fact, its geometric multiplicity is equal to* k*.*

Proof: Recall that we denoted by $\mathcal{N}_k$ the set of matrices $Y \in \mathbb{C}^{n_2 \times n_2}$ such that the matrix G_Y is k-derogatory.

Let us call

$$\mathcal{C} := \{\|Y - D\| : \ Y \in \mathcal{N}_k\}$$

and

$$\mathcal{C}_\lambda := \{\|Y - D\| : \ Y \in \mathcal{N}_{k,\lambda}\}$$

for each $\lambda \in \Omega_k$. Then, by (2.3)

$$\mathcal{C} = \bigcup_{\lambda \in \Omega_k} \mathcal{C}_\lambda .$$

Because 0 is a lower bound of $\mathcal{C}$ and of $\mathcal{C}_\lambda$ for each $\lambda \in \Omega_k$, by [5, Proposition 2.3.6] we have

$$\inf \mathcal{C} = \inf(\bigcup_{\lambda \in \Omega_k} \mathcal{C}_\lambda) = \inf_{\lambda \in \Omega_k} (\inf \mathcal{C}_\lambda) . \tag{4.5}$$

Moreover, for all $\lambda \in \Omega_k$

$$\inf \mathcal{C}_\lambda = \min_{Y \in \mathcal{N}_{k,\lambda}} \|Y - D\|, \tag{4.6}$$

since $\mathcal{N}_{k,\lambda}$ is a closed set (due to the lower semicontinuity of the function $X \mapsto \mathrm{rank}(X)$) .

On the other hand, by Theorem 1.1, for each λ in Ω_k,

$$\sigma_{p_k(\lambda)+1}(S_1(\lambda, D)) =$$

$$\min_{X \in \mathcal{L}_k(\lambda)} \left\| \left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & \lambda I_{n_2} - D \end{array} \right) - \left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & X \end{array} \right) \right\|, \qquad (4.7)$$

where

$$\mathcal{L}_k(\lambda) := \{X \in \mathbb{C}^{n_2 \times n_2} : \operatorname{rank} \left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & X \end{array} \right) \leq n - k\}.$$

If $X \in \mathbb{C}^{n_2 \times n_2}$ is any matrix such that

$$\operatorname{rank} \left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & X \end{array} \right) \leq n - k$$

and we define $X' := \lambda I_{n_2} - X$, then $X' \in \mathcal{N}_{k,\lambda}$; conversely, if $X' \in \mathcal{N}_{k,\lambda}$ and $X := \lambda I_{n_2} - X'$, then

$$\operatorname{rank} \left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & X \end{array} \right) \leq n - k.$$

Consequently, for each $\lambda \in \Omega_k$, by virtue of (4.7),

$$\begin{aligned}
&\sigma_{p_k(\lambda)+1}(S_1(\lambda, D)) = \\
&\min_{X' \in \mathcal{N}_{k,\lambda}} \left\| \left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & \lambda I_{n_2} - D \end{array} \right) - \left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & \lambda I_{n_2} - X' \end{array} \right) \right\| \\
&= \min_{X' \in \mathcal{N}_{k,\lambda}} \left\| \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & (\lambda I_{n_2} - D) - (\lambda I_{n_2} - X') \end{array} \right) \right\| \\
&= \min_{X' \in \mathcal{N}_{k,\lambda}} \|X' - D\| . \qquad (4.8)
\end{aligned}$$

From (4.5), (4.6) and (4.8) we deduce

$$\begin{aligned}
\min_{Y \in \mathcal{N}_k} \|Y - D\| &= \inf_{\lambda \in \Omega_k} \sigma_{p_k(\lambda)+1}\big(S_1(\lambda, D)\big) \\
&= \inf_{\lambda \in \Omega_k} h_k(\lambda) = \min_{\lambda \in \Omega_k} h_k(\lambda).
\end{aligned}$$

Now let $\lambda_0 \in \Omega_k$ be such that

$$h_k(\lambda_0) = \min_{\lambda \in \Omega_k} h_k(\lambda). \qquad (4.9)$$

Let $\tau_1, \ldots, \tau_{n_2}$ be the singular values of $S_1(\lambda_0, D)$ in nonincreasing order. By the singular value decomposition theorem, there exist unitary matrices $U, V \in \mathbb{C}^{n_2 \times n_2}$ such that

$$U^* S_1(\lambda_0, D) V = \operatorname{diag}(\tau_1, \ldots, \tau_{p_k(\lambda_0)}, \tau_{p_k(\lambda_0)+1}, \ldots, \tau_{n_2}).$$

By definition of h_k, see (3.2), we have

$$h_k(\lambda_0) = \sigma_{p_k(\lambda_0)+1}\big(S_1(\lambda_0, D)\big) = \tau_{p_k(\lambda_0)+1}. \tag{4.10}$$

Next we define

$$Y_1 := D + U \operatorname{diag}(0, \ldots, 0, \tau_{p_k(\lambda_0)+1}, \ldots, \tau_{n_2}) V^*. \tag{4.11}$$

As the spectral norm is unitarily invariant it follows that

$$\|Y_1 - D\| = \|\operatorname{diag}(0, \ldots, 0, \tau_{p_k(\lambda_0)+1}, \ldots, \tau_{n_2})\| = \tau_{p_k(\lambda_0)+1}. \tag{4.12}$$

Still it remains to prove that $Y_1 \in \mathcal{N}_k$. In fact, we are going to prove that $Y_1 \in \mathcal{N}_{k,\lambda_0}$. Indeed, calling $\Delta_0 := \operatorname{diag}(0, \ldots, 0, \tau_{p_k(\lambda_0)+1}, \ldots, \tau_{n_2})$,

$$\operatorname{rank}\left(\begin{array}{c|c} \lambda_0 I_{n_1} - A & -B \\ \hline -C & \lambda_0 I_{n_2} - Y_1 \end{array}\right)$$
$$= \operatorname{rank}\left(\begin{array}{c|c} \lambda_0 I_{n_1} - A & -B \\ \hline -C & \lambda_0 I_{n_2} - D - U\Delta_0 V^* \end{array}\right) = n - k,$$

because, by (1.3), subtracting $U\Delta_0 V^*$ to the matrix $\lambda_0 I_{n_2} - D$ we attain to lower the rank of the matrix

$$\left(\begin{array}{c|c} \lambda_0 I_{n_1} - A & -B \\ \hline -C & \lambda_0 I_{n_2} - D \end{array}\right)$$

to the value

$$\operatorname{rank}\left(\begin{array}{c|c} \lambda_0 I_{n_1} - A & -B \\ \hline -C & \lambda_0 I_{n_2} - D - U\Delta_0 V^* \end{array}\right) = n - k.$$

□

Remark 4.1 From (4.2) of Theorem 4.1 we deduce that the nonnegative integer ℓ that appears in (3.9) must be equal to 0: if $\ell > 0$, upper bound (3.10) should imply that

$$\lim_{\lambda\to\lambda_0} \sigma_{n_2-k+1}\big(F(\lambda)\big) = 0;$$

hence

$$\inf_{\lambda\in\Omega_k} h_k(\lambda) = 0,$$

and, by (4.2),

$$\inf_{\substack{Y \in \mathbb{C}^{n_2 \times n_2} \\ i(G_Y) \geq k}} \|Y - D\| = 0;$$

but this contradicts that the set $\mathcal{M}_k$ is open.

Theorem 4.1 can be proved with inf instead of min.

Scalar matrices. Case $k = n$

Following the exposed material in the subsection "Scalar matrices" at the end of Section 2, let

$$\begin{pmatrix} \alpha I_{n_1} & 0 \\ 0 & D \end{pmatrix}$$

be an $n \times n$ matrix with $\alpha \in \mathbb{C}$, $D \in \mathbb{C}^{n_2 \times n_2}$, $n = n_1 + n_2$, and $i \begin{pmatrix} \alpha I_{n_1} & 0 \\ 0 & D \end{pmatrix} < n$. Consider the problem of finding the nearest matrix $Y \in \mathbb{C}^{n_2 \times n_2}$ to D so that $i \begin{pmatrix} \alpha I_{n_1} & 0 \\ 0 & Y \end{pmatrix} \geq n$; or what is the same, with $\begin{pmatrix} \alpha I_{n_1} & 0 \\ 0 & Y \end{pmatrix}$ a scalar matrix. Now we are going to see that the conclusions (4.2) and (4.3) of Theorem 4.1 follows straightforwardly in this case. Taking into account (2.16) the domain of the function h_n is $\Omega_n = \{\alpha\}$. Furthermore $\rho_1(\alpha) = 0$ and, accordingly, $p_n(\alpha) = 0$; hence

$$h_n(\alpha) = \sigma_1(\alpha I_{n_2} - D) = \|\alpha I_{n_2} - D\|.$$

and

$$\min_{\lambda \in \Omega_n} h_n(\lambda) = h_n(\alpha) = \|\alpha I_{n_2} - D\|. \tag{4.13}$$

On the other hand,

$$\min_{\substack{Y \in \mathbb{C}^{n_2 \times n_2} \\ G_Y \text{ scalar matrix}}} \|Y - D\| = \|\alpha I_{n_2} - D\|, \tag{4.14}$$

because the set

$$\{Y \in \mathbb{C}^{n_2 \times n_2} \mid G_Y \text{ is a scalar matrix}\}$$

only has an element: αI_{n_2}. Therefore, from (4.13) and (4.14) the assertion (4.2) is evident in this case. Besides, by (4.3) and (4.4), as $S_1(\alpha, D) := \alpha I_{n_2} - D$, let $U, V \in \mathbb{C}^{n_2 \times n_2}$ be the unitary matrices that appear in the singular value decomposition of $\alpha I_{n_2} - D$:

$$U^*(\alpha I_{n_2} - D)V = \operatorname{diag}(\tau_1, \ldots, \tau_{n_2}) \qquad \text{with } \tau_1 > 0.$$

Take

$$Y_1 := D + U\,\mathrm{diag}(\tau_1, \ldots, \tau_{n_2})V^* = D + \alpha I_{n_2} - D = \alpha I_{n_2};$$

which confirms the aforementioned exposed.

5. k-Derogatory pseudospectrum

Let $M \in \mathbb{C}^{n\times n}$; we will denote by $S_k(M)$ the set of k-derogatory eigenvalues of M. So, $S_1(M) = \Lambda(M)$, the spectrum of M. Let $G \in \mathbb{C}^{n\times n}$ be the partitioned matrix

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with $A \in \mathbb{C}^{n_1\times n_1}, B \in \mathbb{C}^{n_1\times n_2}, C \in \mathbb{C}^{n_2\times n_1}$, $D \in \mathbb{C}^{n_2\times n_2}$, and $i(G) < k$.

Where are the k-derogatory eigenvalues of all matrices

$$G_Y := \begin{pmatrix} A & B \\ C & Y \end{pmatrix}$$

such that $Y \in \mathbb{C}^{n_2\times n_2}$, is sufficiently close to D? This question is closely related with the problem treated in Section 4. We would like to find out the geometric description of the set in the complex plane formed by the k-derogatory eigenvalues of all the matrices G_Y whose distance from G is less than or equal to a prefixed $\varepsilon > 0$. If ε is less than

$$\min_{\lambda\in\Omega_k} h_k(\lambda),$$

then there is no k-derogatory eigenvalue of the matrices G_Y where $\|Y - D\| \leq \varepsilon$, because, by (4.2), all these matrices satisfy $i(G_Y) < k$. So, a necessary condition for the set

$$\bigcup_{\|Y-D\|\leq\varepsilon} S_k(G_Y) \quad (k\text{-derogatory pseudospectrum of } G \text{ of radius } \varepsilon)$$

to be nonempty is that

$$\varepsilon \geq \min_{\lambda\in\Omega_k} h_k(\lambda).$$

It is natural that the k-derogatory pseudospectrum of G of radius ε is equal to the set enclosed by the ε–level curve of the function $f(x, y) := h_k(x+yi)$. This fact is consequence of the following theorem.

Theorem 5.1 *With the preceding notations, let $\varepsilon > 0$ be a real number. Then*

$$\bigcup_{\substack{Y\in\mathbb{C}^{n_2\times n_2}\\ \|Y-D\|\leq\varepsilon}} S_k(G_Y) = \{z \in \Omega_k : h_k(z) \leq \varepsilon\}. \tag{5.1}$$

Proof: Recall that

$$\Omega_k = \{\lambda \in \mathbb{C} : \rho_1(\lambda) \leq n - k\} = \{\lambda \in \mathbb{C} : \mathcal{N}_{k,\lambda} \neq \emptyset\}.$$

Let $z \in \Omega_k$ be such that $h_k(z) \le \varepsilon$; then

$$\sigma_{p_k(z)+1}\big(S_1(z,D)\big) \le \varepsilon. \tag{5.2}$$

But

$$\sigma_{p_k(z)+1}\big(S_1(z,D)\big) =$$

$$\min_{X \in \mathcal{L}_k(z)} \left\| \left(\begin{array}{c|c} zI_{n_1} - A & -B \\ \hline -C & zI_{n_2} - D \end{array} \right) - \left(\begin{array}{c|c} zI_{n_1} - A & -B \\ \hline -C & X \end{array} \right) \right\|$$

$$= \min_{X' \in \mathcal{N}_{k,z}} \|X' - D\| = \|X_0' - D\|,$$

where

$$\mathcal{L}_k(z) := \{X \in \mathbb{C}^{n_2 \times n_2} : \operatorname{rank} \left(\begin{array}{c|c} zI_{n_1} - A & -B \\ \hline -C & X \end{array} \right) \le n - k\}.$$

with $X_0' \in \mathcal{N}_{k,z}$ (what implies z is a k-derogatory eigenvalue of $G_{X_0'}$).

Furthermore, from (5.2) we have $\|X_0' - D\| \le \varepsilon$. Hence

$$\{z \in \Omega_k : h_k(z) \le \varepsilon\} \subset \bigcup_{\|Y-D\| \le \varepsilon} S_k(G_Y).$$

Reciprocally, if $z \in S_k(G_{X_0'})$ for some $X_0' \in \mathbb{C}^{n_2 \times n_2}$ such that $\|X_0' - D\| \leq \varepsilon$, it follows that $X_0' \in \mathcal{N}_{k,z}$; this implies $\mathcal{N}_{k,z} \neq \emptyset$, so $z \in \Omega_k$. Besides,

$$\|X_0' - D\| = \left\| \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & (zI_{n_2} - D) - (zI_{n_2} - X_0') \end{array} \right) \right\| =$$

$$\left\| \left(\begin{array}{c|c} zI_{n_1} - A & -B \\ \hline -C & zI_{n_2} - D \end{array} \right) - \left(\begin{array}{c|c} zI_{n_1} - A & -B \\ \hline -C & zI_{n_2} - X_0' \end{array} \right) \right\| \geq$$

$$\sigma_{p_k(z)+1}\big(S_1(z, D)\big) = h_k(z);$$

therefore it implies $\varepsilon \geq h_k(z)$. Hence

$$\bigcup_{\|Y - D\| \leq \varepsilon} S_k(G_Y) = \{z \in \Omega_k : h_k(z) \leq \varepsilon\}. \qquad \square$$

Structured pseudospectrum

Let $G \in \mathbb{C}^{n\times n}$ be the partitioned matrix

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with $A \in \mathbb{C}^{n_1\times n_1}, B \in \mathbb{C}^{n_1\times n_2}, C \in \mathbb{C}^{n_2\times n_1}$ and $D \in \mathbb{C}^{n_2\times n_2}$. Here G is *any matrix*, and it is not necessary that $i(G) < k$.

Where are the eigenvalues of all matrices

$$G_Y := \begin{pmatrix} A & B \\ C & Y \end{pmatrix}$$

such that $Y \in \mathbb{C}^{n_2\times n_2}$ is sufficiently close to D? This question is closely related with the problem treated in the first part of this Section 5. We would like to find out the geometric description of the set in the complex plane formed by the eigenvalues of all the matrices G_Y whose distance from G is less than or equal to a prefixed $\varepsilon > 0$. The same question, if it is permitted to perturb in all entries of the matrix G, has been studied in [16], [17], with the name of pseudospectrum of the matrix G of radius $\varepsilon > 0$

$$\bigcup_{\substack{G' \in \mathbb{C}^{n\times n} \\ \|G'-G\| \le \varepsilon}} \Lambda(G').$$

It was proved that

$$\bigcup_{\substack{G' \in \mathbb{C}^{n\times n} \\ \|G'-G\| \le \varepsilon}} \Lambda(G') = \{z \in \mathbb{C} : \sigma_n(zI_n - G) \le \varepsilon\},$$

where $\sigma_n(zI_n - G)$ is the minimum singular value of the matrix $zI_n - G$.

For every $\lambda \in \mathbb{C}$, define

$$\mathcal{N}_{1,\lambda} := \left\{ Y \in \mathbb{C}^{n_2 \times n_2} : \lambda \text{ is an eigenvalue of } \begin{pmatrix} A & B \\ C & Y \end{pmatrix} \right\}$$

and let Ω_1 be the set $\{\lambda \in \mathbb{C} : \mathcal{N}_{1,\lambda} \neq \emptyset\}$. Given the matrices A, B and C, can it happen that for some $\lambda \in \mathbb{C}$ the set $\mathcal{N}_{1,\lambda}$ be empty? The answer is affirmative as we can see that for all $y \in \mathbb{C}$

$$\det \begin{pmatrix} \lambda - 1 & -2 \\ -3 & \lambda - y \end{pmatrix} = (\lambda - 1)(\lambda - y) - 6;$$

So if $\lambda = 1$, there is no y such that 1 be an eigenvalue of

$$\left(\begin{array}{c|c} 1 & 2 \\ \hline 3 & y \end{array}\right).$$

In fact, in this example $\Omega_1 = \mathbb{C} \setminus \{1\}$. Calling for any $\lambda \in \Omega_1$, $\rho_1(\lambda)$ as in (2.4) and $S_1(\lambda, D)$ as in (2.7), it is simple to see that $\Omega_1 = \{\lambda \in \mathbb{C} : \rho_1(\lambda) \leq n - 1\}$. We have always $\mathbb{C} \setminus \Lambda(A) \subset \Omega_1$, because for all $\lambda \in \mathbb{C} \setminus \Lambda(A)$ it follows $\rho_1(\lambda) = n_1$ and $n_1 \leq n - 1$.

Now we define the set $\Omega^{(1)} := \{\lambda \in \mathbb{C} : n_1 \leq \rho_1(\lambda) \leq n - 1\}$. Obviously $\Omega^{(1)} \subset \Omega_1$, but the content can be strict. For example, given the matrix

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left(\begin{array}{ccc|cc} 3 & 1 & -2 & 0 & 0 \\ 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ \hline 0 & 0 & 0 & 6 & -4 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right),$$

we have that $3 \in \Omega_1$, because 3 is an eigenvalue of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$; but $\rho_1(3) = 2 \not\geq 3$, so $3 \notin \Omega^{(1)}$.

For all $\lambda \in \Omega^{(1)}$, we define $p_1(\lambda) := n - 1 - \rho_1(\lambda)$ and the function

$$h_1 : \Omega^{(1)} \to \mathbb{R}$$

by $h_1(\lambda) := \sigma_{p_1(\lambda)+1}\big(S_1(\lambda, D)\big)$. It is easy to see that this has meaning given that for all $\lambda \in \Omega^{(1)}$, $0 \leq p_1(\lambda) \leq n_2 - 1$. Moreover, $\lambda \in \Lambda(G) \cap \Omega^{(1)}$ if and only if $h_1(\lambda) = 0$. By an analogous way of the proof of Theorem 5.1 we can prove the following result.

Theorem 5.2 *Let $G \in \mathbb{C}^{n \times n}$ be the partitioned matrix*

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with $A \in \mathbb{C}^{n_1 \times n_1}, B \in \mathbb{C}^{n_1 \times n_2}, C \in \mathbb{C}^{n_2 \times n_1}$ and $D \in \mathbb{C}^{n_2 \times n_2}$. And let $\varepsilon > 0$ be a real number. Then

$$\bigcup_{\substack{Y \in \mathbb{C}^{n_2 \times n_2} \\ \|Y - D\| \leq \varepsilon}} \Lambda(G_Y) = \{z \in \Omega^{(1)} : h_1(z) \leq \varepsilon\} \cup \Lambda(G).$$

There is an alternative characterization of the restricted pseudospectrum of G of radius $\varepsilon > 0$

$$\bigcup_{\substack{Y \in \mathbb{C}^{n_2 \times n_2} \\ \|Y - D\| \leq \varepsilon}} \Lambda(G_Y),$$

as

$$\bigcup_{\substack{Y \in \mathbb{C}^{n_2 \times n_2} \\ \|Y - D\| \leq \varepsilon}} \Lambda(G_Y) = \left\{ z \in \mathbb{C} \setminus \Lambda(G) : \sigma_{n_2}\big(R(z)\big) \leq \varepsilon \right\} \cup \Lambda(G),$$

where

$$R(z) := \left[(0, I_{n_2})\,(zI_n - G)^{-1} \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} \right]^{\dagger}$$

is the Moore-Penrose inverse of a transfer matrix. See [8, Proposition 2.3, p. 128].

6. Conclusions

In [18] it was reformulated a result of [4] that gives in a precise manner how to find the nearest matrix $\begin{pmatrix} A & B \\ C & Y \end{pmatrix}$ that lowers the rank of the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, by means of ordinary singular values of a matrix related with A, B, C and D through the Moore-Penrose inverse. Given that many important features of the Jordan canonical form of a matrix (in particular, the geometric multiplicity of its eigenvalues) can be formulated in terms of ranks of certain matrices, we have been able to obtain a solution to related nearness matrix problems from this theorem.

We have obtained the nearest matrix $\begin{pmatrix} A & B \\ C & Y \end{pmatrix}$, $i\begin{pmatrix} A & B \\ C & Y \end{pmatrix} \geq k$, to the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that $i\begin{pmatrix} A & B \\ C & D \end{pmatrix} < k$, if we perturb only in D. Also, we have established the relation of this last problem with the question of where are the k-derogatory eigenvalues of matrices $\begin{pmatrix} A & B \\ C & Y \end{pmatrix}$ with Y adequately close to D.

References

[1] H. Baumgärtel: *Analytic Perturbation Theory for Matrices and Operators*. Birkhäuser, 1985. 41

[2] R. Bhatia: *Matrix Analysis*. Springer, 1997.

[3] S.L. Campbell, C.D. Meyer: *Generalized Inverses of Linear Transformations*. Pitman, London, 1979. 7, 8, 26

[4] J.W. Demmel: The smallest perturbation of a submatrix which lowers the rank and constrained total least squares problems. *SIAM J. Numer. Anal.* **24**,(1) (1987) 199–206. 7, 8, 64

[5] J. Dieudonné: *Fondements de l'Analyse Moderne*. Gauthier-Villars, Paris, 1966. 47

[6] I. Gohberg, P. Lancaster, L. Rodman: *Invariant Subspaces of Matrices with Applications*. Wiley, 1986. 33

[7] J.M. Gracia, I. de Hoyos, F.E. Velasco: Safety neighbourhoods for the invariants of the matrix similarity. *Linear Multilinear Algebra* **46**, (1999) 25–49. 5, 39

[8] D. Hinrichsen, B. Kelb: Spectral value sets: a graphical tool for robustness analysis. *Systems and Control Letters* **21**, (1993) 127–136. 63

[9] R. A. Horn, C. R. Johnson: *Topics in Matrix Analysis*. Cambridge University Press, 1991. 39

[10] V. Ionescu, C. Oară, M. Weiss: *Generalized Riccati Theory and Robust Control. A Popov Function Approach*. Wiley, 1999. 33

[11] T. Kato: *A Short Introduction to Perturbation Theory for Linear Operators*. Springer, 1982. 41, 43

[12] P. Lancaster, M. Tismenetsky: *The Theory of Matrices with Applications*. Academic Press, 1985. 32, 43

[13] G. Marsaglia and G. P. H. Styan: Equalities and inequalities for ranks of matrices. *Linear Multilinear Algebra* **2**, (1974) 269–292. 7, 8, 27

[14] J. F. Queiró and E. Marques de Sá: Singular values and invariant factors of matrix sums and products. *Linear Algebra Appl.* **225**, (1995) 43–56. 31

[15] F. C. Silva: On the number of invariant factors of partially prescribed matrices and control theory. *Linear Algebra Appl.* **311**, (2000) 1–12. 12, 26

[16] L.N. Trefethen: Pseudospectra of matrices, in *Numerical Analysis1991*. D.F. Griffiths and G.A. Watson, eds., Longman Scientific and Technical, Harlow, UK, 1992, pp. 234–266. 11, 59

[17] L.N. Trefethen: Pseudospectra of linear operators. *SIAM Rev.* **39**, (3) (1997) 383–406. 11, 59

[18] M. Wei: Perturbation theory for the Eckart-Young-Mirsky theorem and the constrained total least squares problem. *Linear Algebra Appl.* **280**, (1998) 267–287. 7, 8, 64