Multiplicities of pseudoeigenvalues

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$\begin{pmatrix} I & L \\ A & S \end{pmatrix}$

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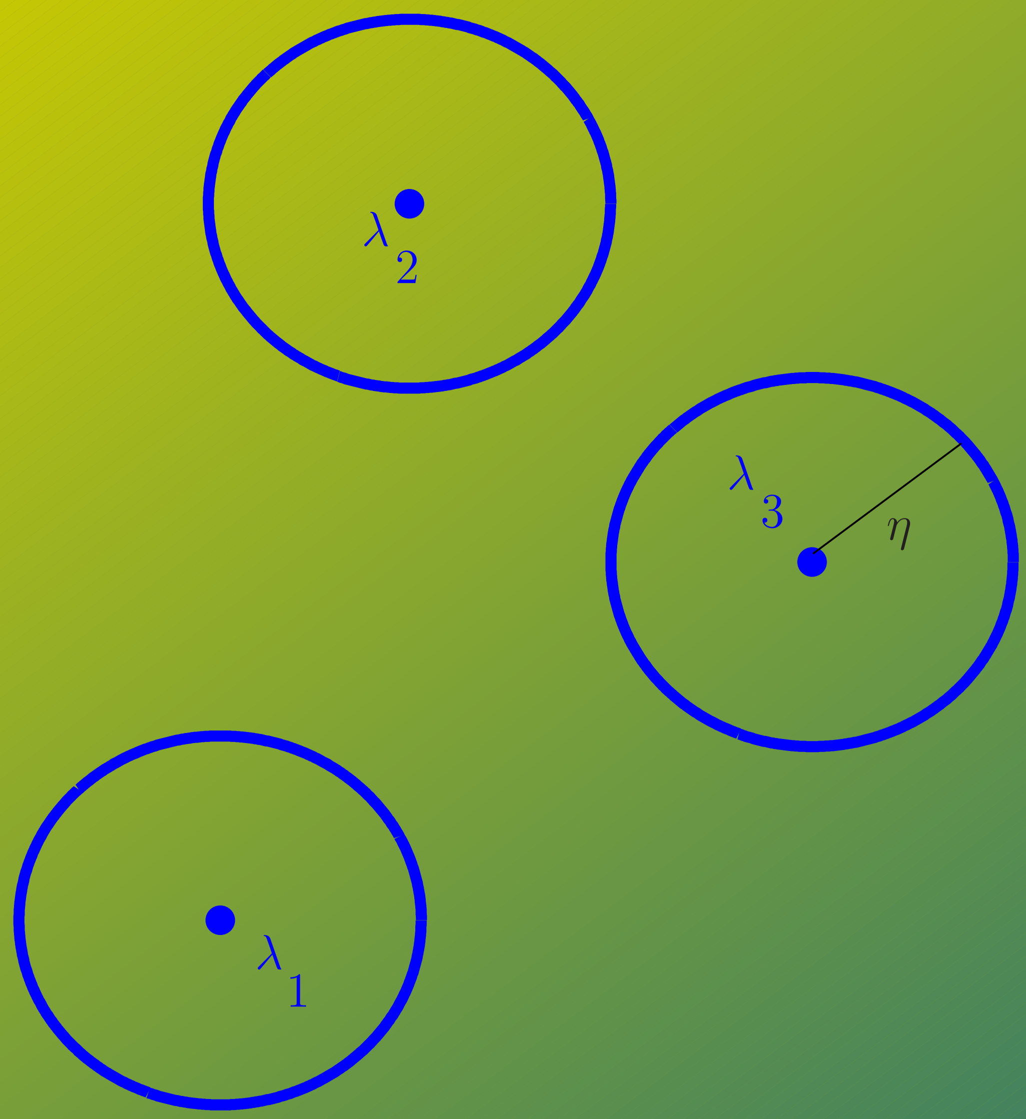
λ_2
λ_3
η
λ_1

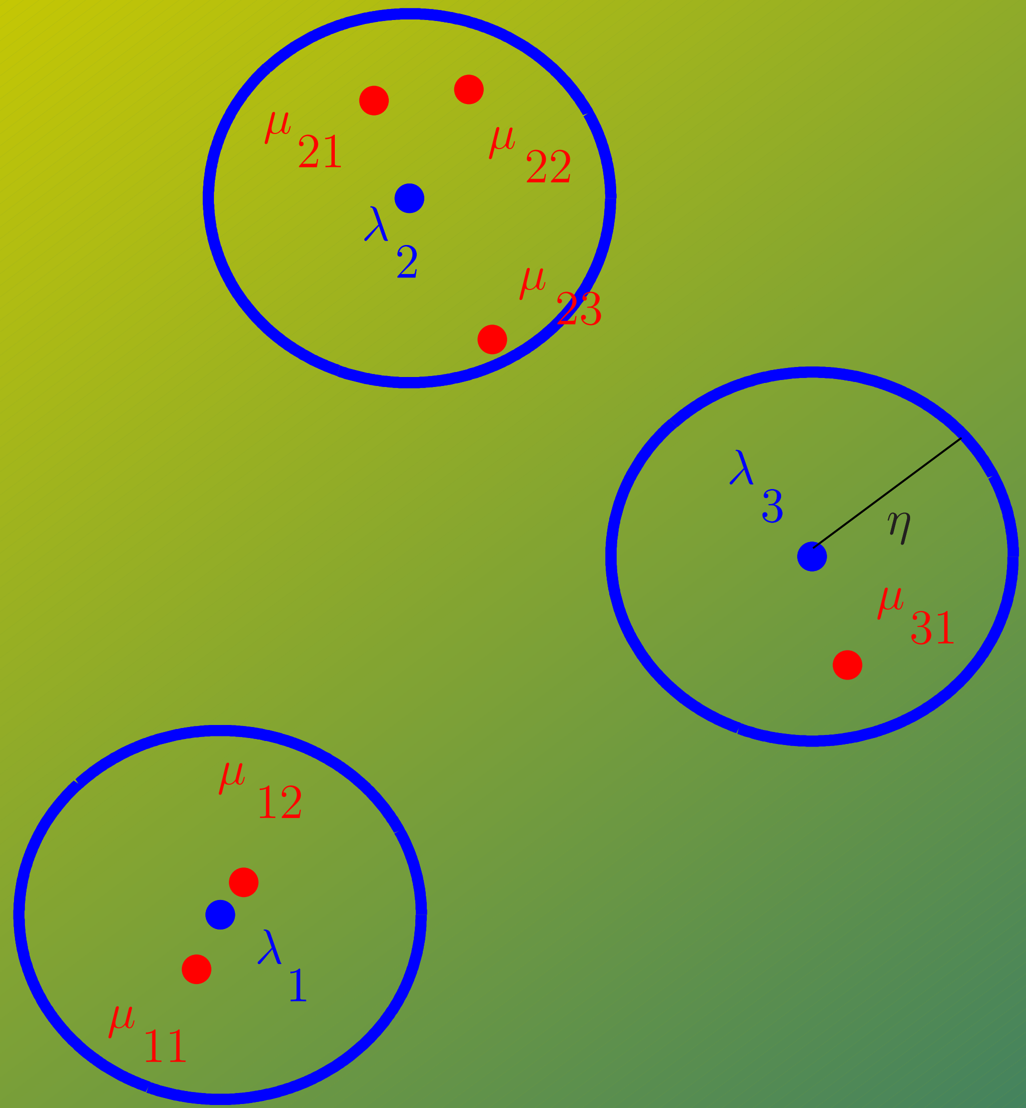

Figure 1: $\|A' - A\| < \delta$

Converse continuity

Fix $\varepsilon > 0$; if $\|X - A\| < \varepsilon$,

- what about $\mathrm{m}(\xi, X)$ and $\mathrm{m}(\alpha, A)$?

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Pseudospectrum of level ε

$$\Lambda_\varepsilon(A) := \text{set of } \varepsilon\text{–pseudoeigenvalues of } A$$

$$\Lambda_\varepsilon(A) = \bigcup_{\substack{X \in \mathbb{C}^{n \times n} \\ \|X - A\| \leq \varepsilon}} \Lambda(X)$$

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- $\Lambda_\varepsilon(A)$ compact set.
- $\Lambda'_\varepsilon(A)$ open set.

Theorem 1. $A \in \mathbb{C}^{n\times n}$, $\varepsilon > 0$; $S_1, \ldots, S_{\rho(\varepsilon,A)}$ connected components of $\Lambda'_\varepsilon(A)$; $\forall X \in \mathbb{C}^{n\times n}$, $\|X - A\| < \varepsilon$, $\forall i = 1, \ldots, \rho(\varepsilon, A)$,

$$\boxed{\sum_{\xi\in\Lambda(X)\cap S_i} \mathrm{m}(\xi, X) = \sum_{\alpha\in\Lambda(A)\cap S_i} \mathrm{m}(\alpha, A).}$$

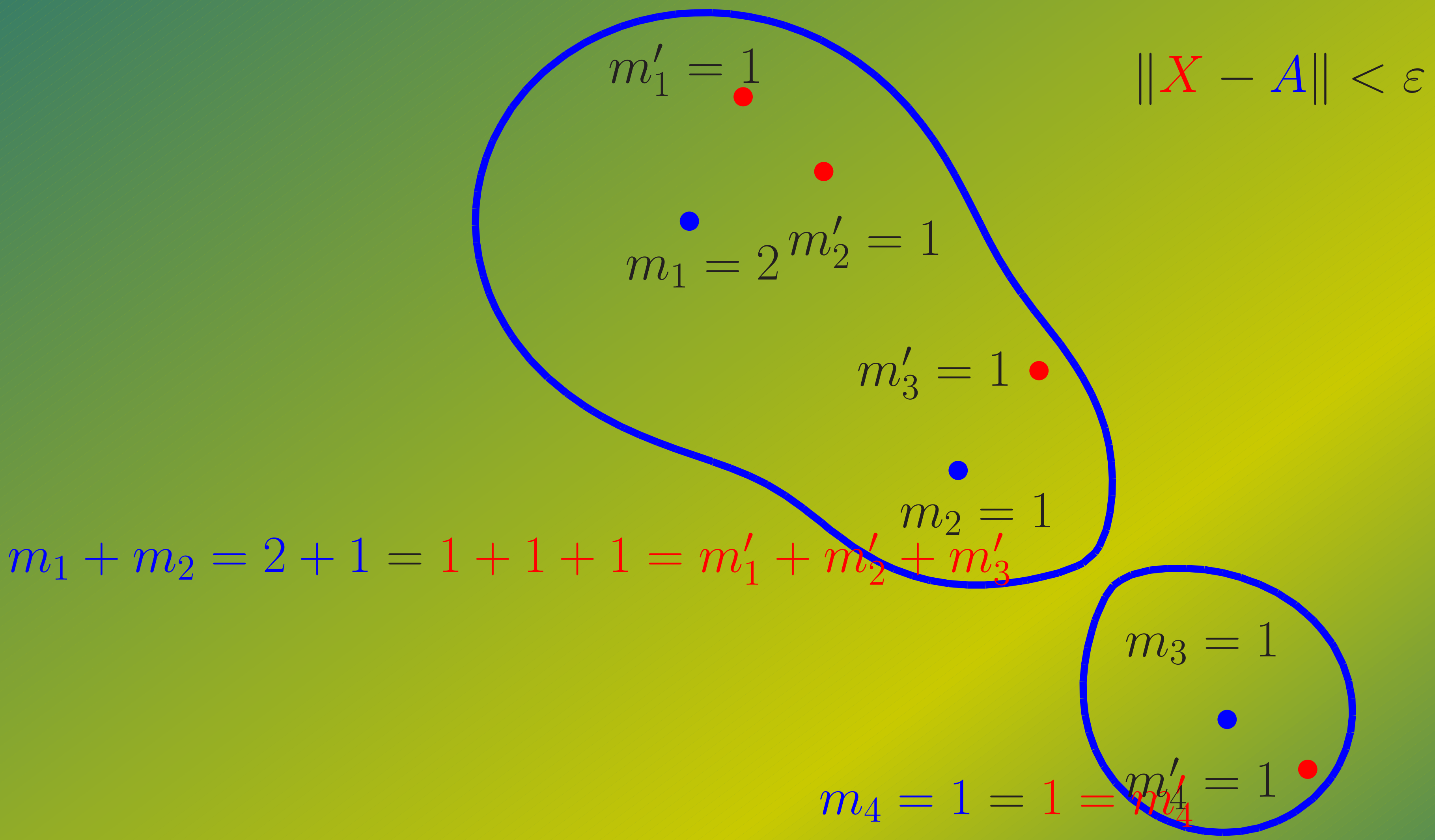

Figure 2: Constant sum

Proof of Theorem 1

$$\|X - A\| < \varepsilon$$

$$Z(t) := A + t(X - A),\ 0 \le t \le 1,$$

$$p_{Z(t)}(\lambda) = \det\big(\lambda I - Z(t)\big);$$

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Sylvester's resultant $(2n-1) \times (2n-1)$

$$R(t) = \mathrm{Res}\big(p_{Z(t)}, p'_{Z(t)}\big)$$

$$\#\Lambda(Z(t)) = n - \nu\big(R(t)\big)$$

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E top. space; $\mathcal{A}\colon E \longrightarrow \mathbb{C}^{n\times n}$ continuous.

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$t_0 \in E$ point of bifurcation of the spectrum of $\mathcal{A}$ if
$\forall\, V(t_0)$ neighbourhood of t_0, $n(t) \not\equiv$ cte.

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- F points bifurcation of spectrum of Z.

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Lemma 3. Constant multiplicities

$(a,b) \subset [0,1] \setminus F,$

- the multiplicities of $\lambda_1(t), \ldots, \lambda_s(t)$ are constant.

Proof. $\#\Lambda(Z(t)) \equiv s$ on (a,b).

- Graphic explanation: figures.pdf

Corollaries of Theorem 1

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$$\Downarrow$$

$$\forall X : \|X - A\| < \varepsilon, \forall \xi \in \Lambda(X), \quad \mathrm{m}(\xi, X) \le \mu(\varepsilon, A).$$

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Lower bound

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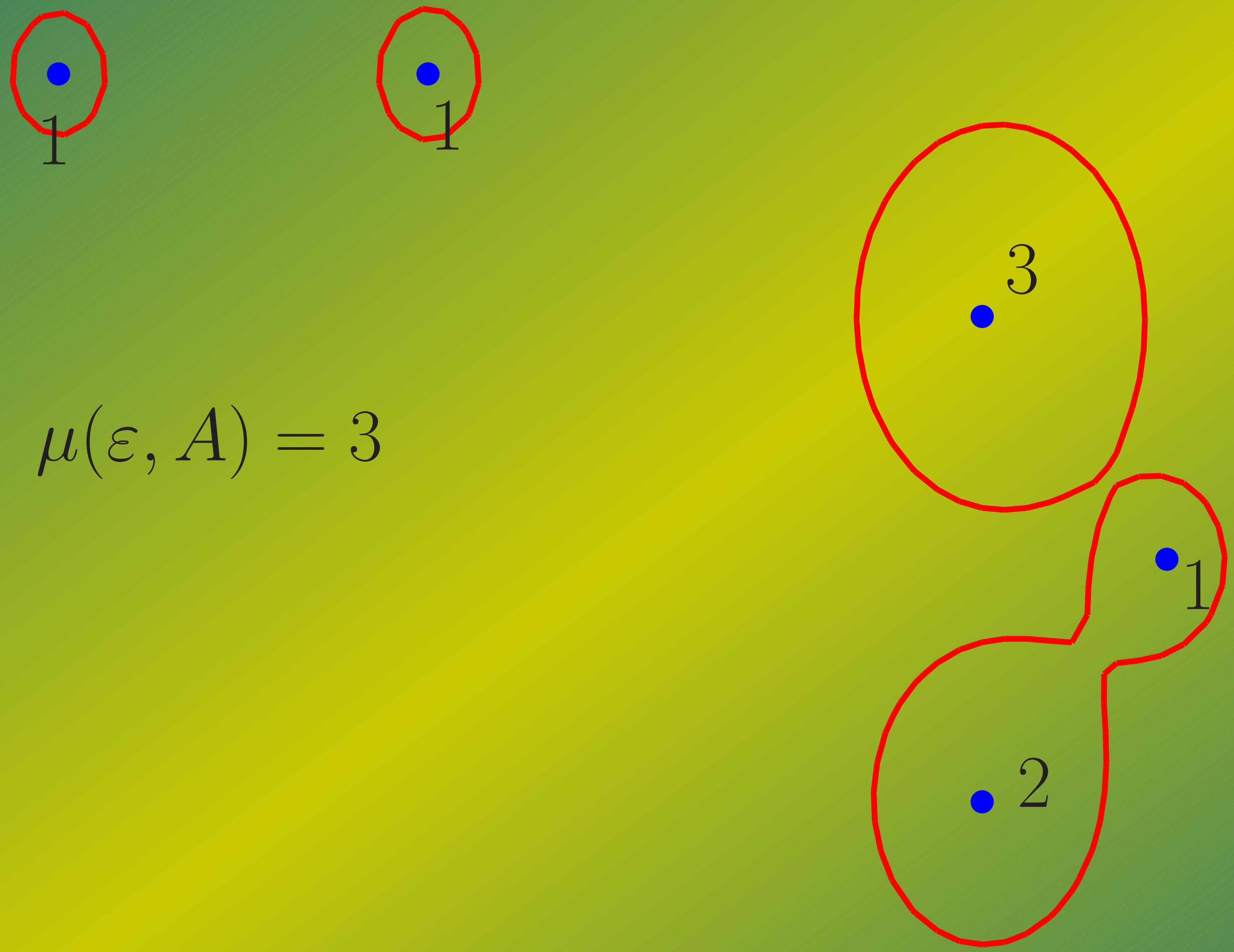
$$\sup\{\varepsilon > 0\colon \mu(\varepsilon, A) = \mathrm{m}(A)\} \leq \min_{\mathrm{m}(X)\geq \mathrm{m}(A)+1} \|X - A\|$$



$$\mu(\varepsilon, A) = 3$$

Figure 3: Greater multiplicity

The End