Multiplicities of pseudoeigenvalues

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$$\begin{bmatrix} I & L \\ A & S \end{bmatrix}$$

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Pseudospectrum of level ε

$\varepsilon > 0, A \in \mathbb{C}^{n\times n}$,

$$\Lambda_\varepsilon(A) = \bigcup_{\substack{X\in\mathbb{C}^{n\times n}\\ \|X-A\|\leq\varepsilon}} \Lambda(X),$$ ε–pseudoeigenvalue of A.

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Strict pseudospectrum of level ε

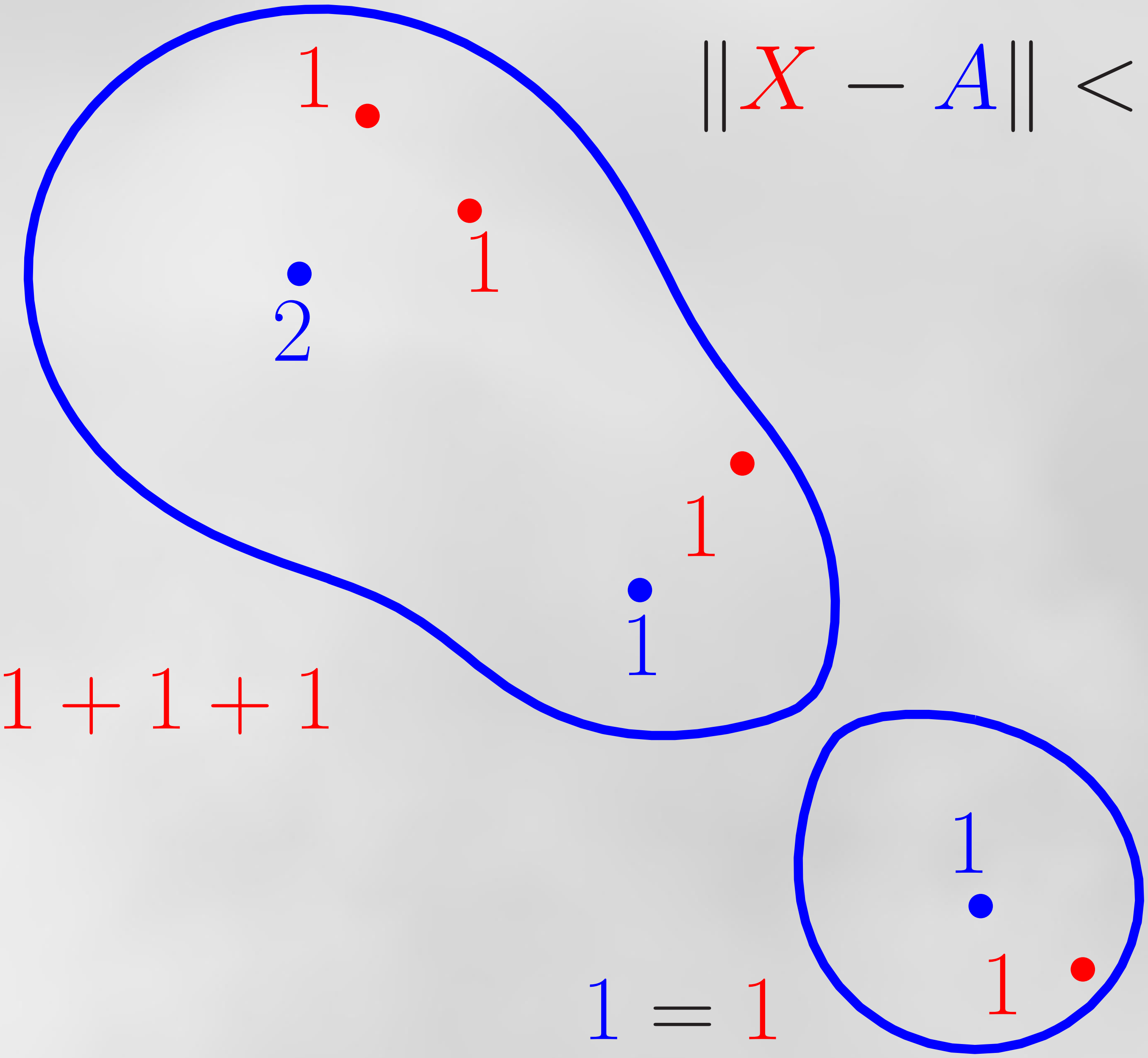
$$\Lambda'_\varepsilon(A) = \bigcup_{\substack{X\in\mathbb{C}^{n\times n}\\ \|X-A\|<\varepsilon}} \Lambda(X)$$

- $\Lambda_\varepsilon(A)$ compact set; $\Lambda'_\varepsilon(A)$ open set.

Theorem 1. $A \in \mathbb{C}^{n\times n}$, $\varepsilon > 0$; $S_1, \ldots, S_{\rho(\varepsilon)}$ connected components of $\Lambda'_\varepsilon(A)$; $\forall X \in \mathbb{C}^{n\times n}$, $\|X - A\| < \varepsilon$, $\forall i = 1, \ldots, \rho(\varepsilon)$,

$$\boxed{\sum_{\xi\in\Lambda(X)\cap S_i} \mathrm{m}(\xi, X) = \sum_{\alpha\in\Lambda(A)\cap S_i} \mathrm{m}(\alpha, A).}$$

$$\|X - A\| < \varepsilon$$



$$2 + 1 = 1 + 1 + 1$$

$$1 = 1$$

Proof of Theorem 1

E top. space; continuous $\mathcal{A}: E \to \mathbb{C}^{n \times n}$;

$N(t) := \#\{\mu \in \mathbb{C} : \det\big(\mu I - \mathcal{A}(t)\big) = 0\},\ t \in E.$

$t_0 \in E$, bifurcation point of the spectrum of $\mathcal{A}$ if N is not constant on every neighbourhood of t_0.

$$Z(t) = A + t(X - A),\ 0 \le t \le 1.$$

- $\exists$ finite $F \subset [0, 1]$,

$$\#\Lambda(Z(t)) = \begin{cases} s & \text{if } t \in [0,1] \setminus F, \\ < s & \text{if } t \in F. \end{cases}$$

- F bifurcation points of spectrum of Z.

Lemma 2. Continuous parametrization

$\exists\ \lambda_j : [0,1] \to \mathbb{C},\ j = 1, \ldots, s,$

$\forall t \in [0,1],\ \{\lambda_1(t), \ldots, \lambda_s(t)\} = \Lambda(Z(t)) \subset \Lambda'_\varepsilon(A).$

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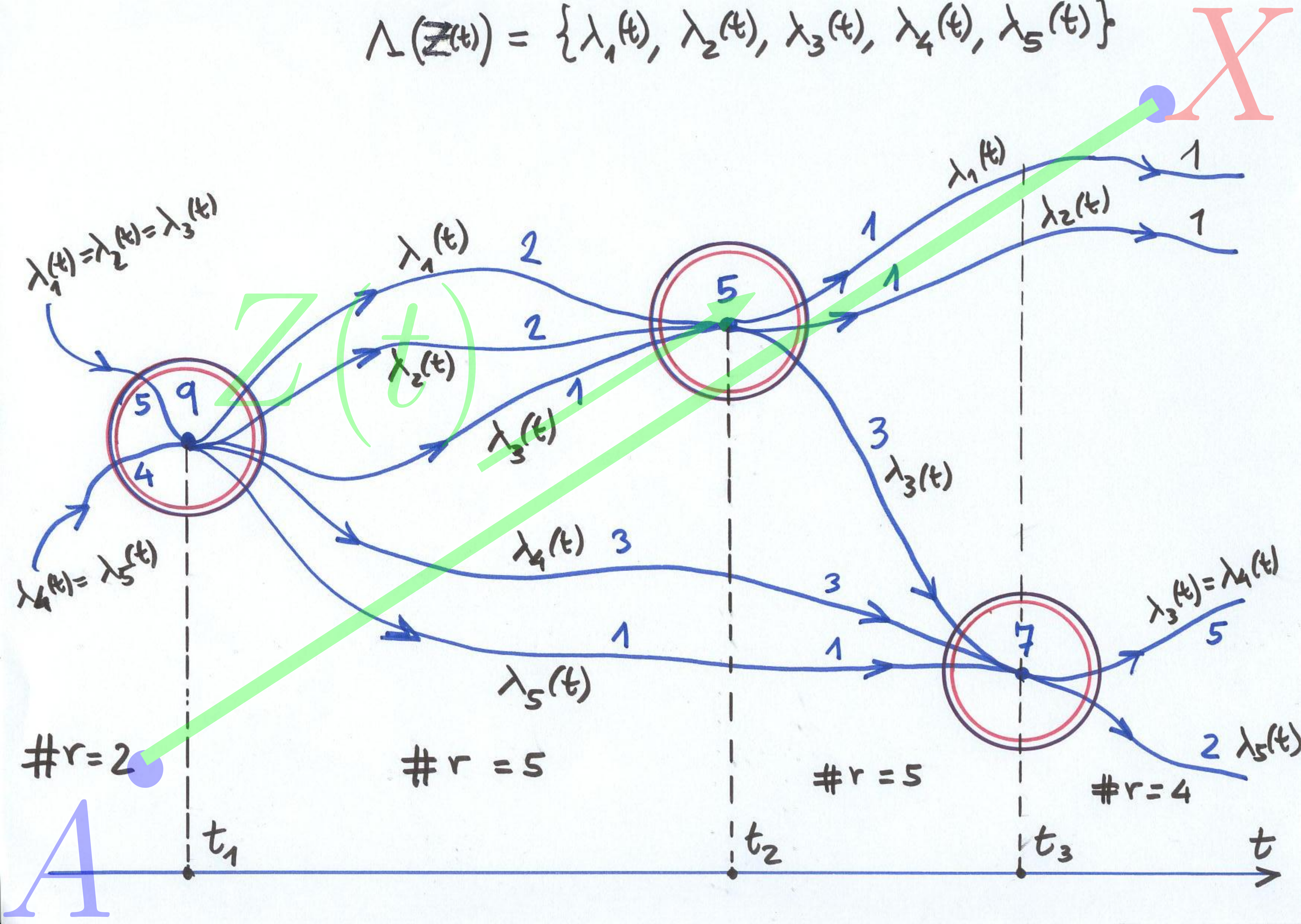
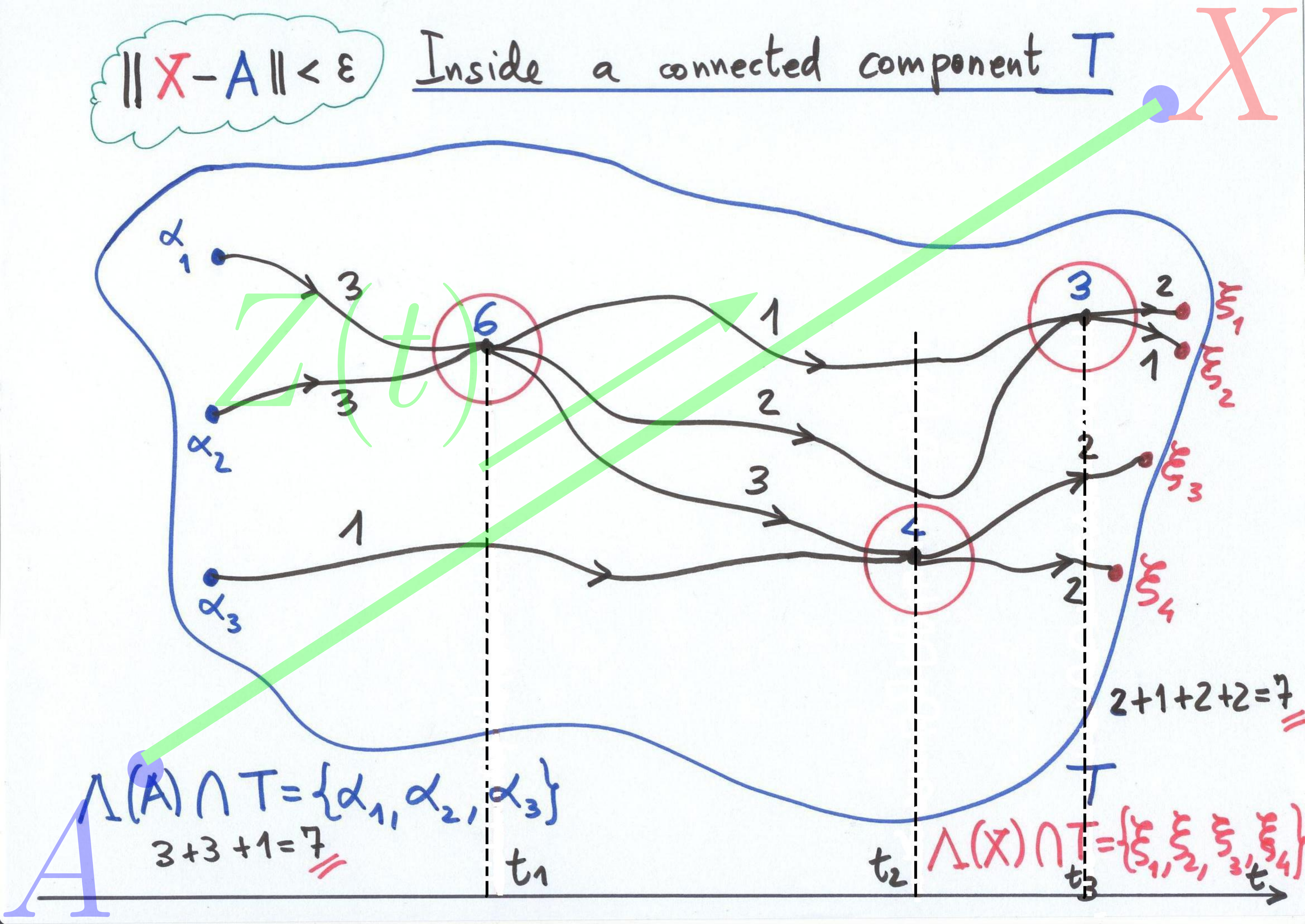
Lemma 3. Constant multiplicities $(a,b) \subset [0,1] \setminus F,$

- the multiplicities of $\lambda_1(t), \ldots, \lambda_s(t)$ are constant.



Λ(Z(t)) = {λ1(t), λ2(t), λ3(t), λ4(t), λ5(t)}
X
Z(t)
λ1(t)=λ2(t)=λ3(t)
λ4(t)= λ5(t)
λ1(t)
λ2(t)
λ3(t)
λ4(t)
λ5(t)
λ3(t)=λ4(t)
9
5
7
#r=2
#r =5
#r =5
#r=4
t1
t2
t3
t
A



||X−A|| < ε
Inside a connected component T
X
Z(t)
α1
α2
α3
3
3
1
6
1
2
3
3
4
2
1
2
2
ξ1
ξ2
ξ3
ξ4
2+1+2+2=7
T
Λ(A) ∩ T = {α1, α2, α3}
3+3+1=7
t1
t2
t3
t
Λ(X) ∩ T = {ξ1, ξ2, ξ3, ξ4}
A

Lower bound

$$\mu(\varepsilon, A) := \max_{1 \le i \le \rho(\varepsilon)} \sum_{\alpha \in \Lambda(A) \cap S_i} \mathrm{m}(\alpha, A),$$

$$\mathrm{m}(A) := \max_{\alpha \in \Lambda(A)} \mathrm{m}(\alpha, A), \quad \mathrm{m}(A) \le k \le n,$$

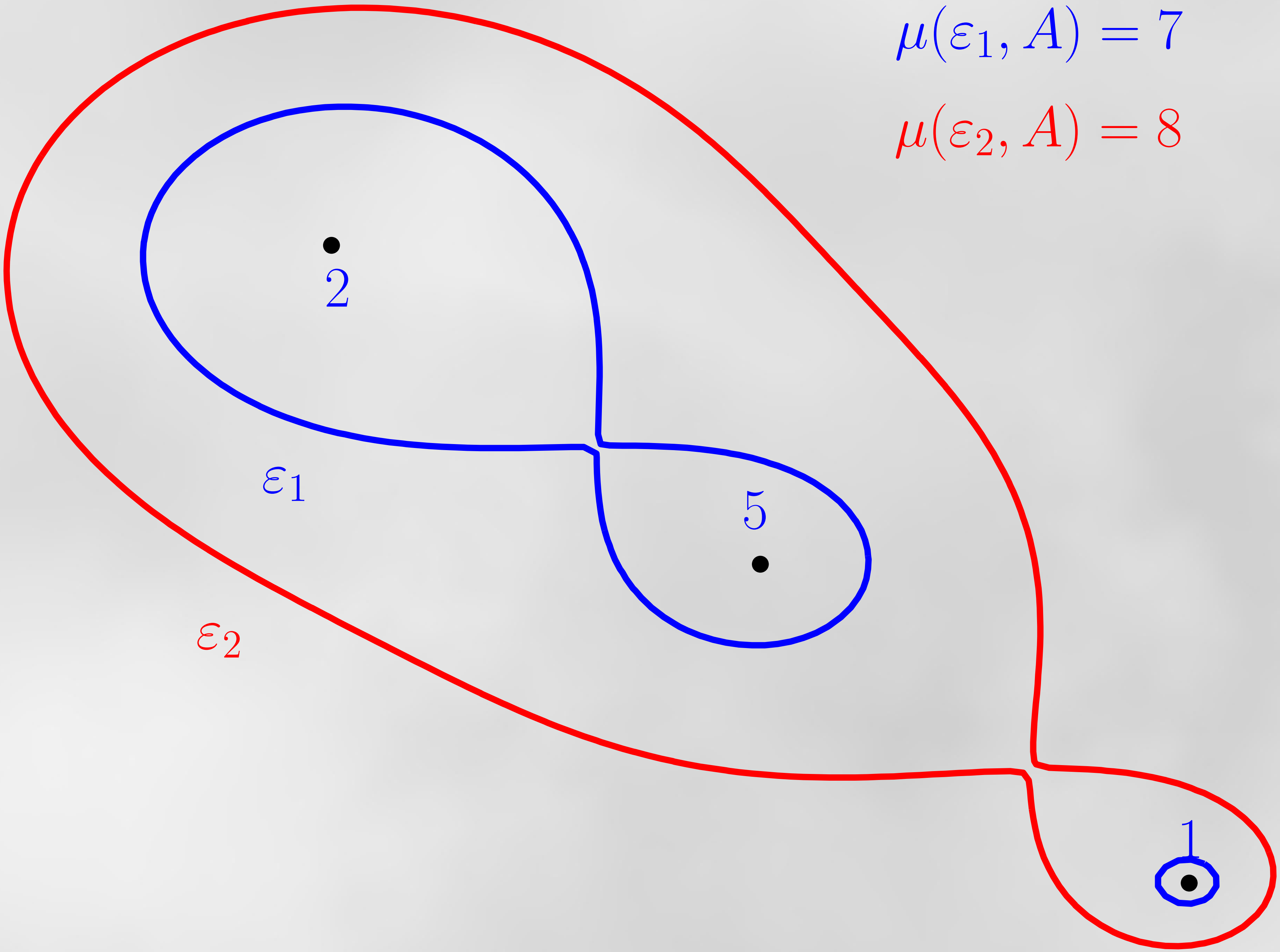
$$\Downarrow$$

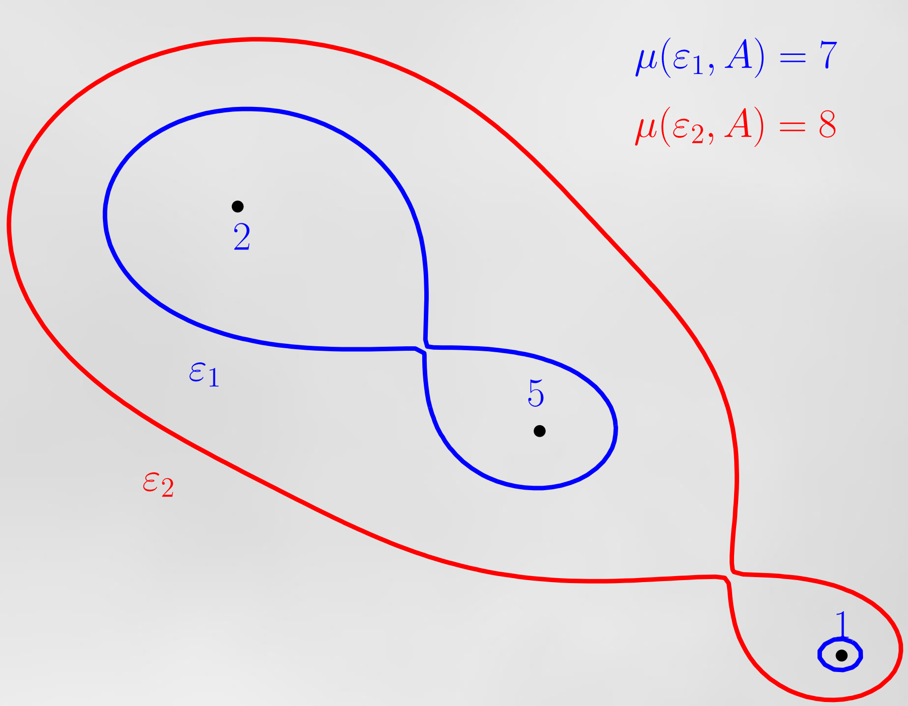
$$\sup \{\varepsilon > 0 \colon \mu(\varepsilon, A) \le k\} \le \min_{\mathrm{m}(X) \ge k+1} \|X - A\|.$$

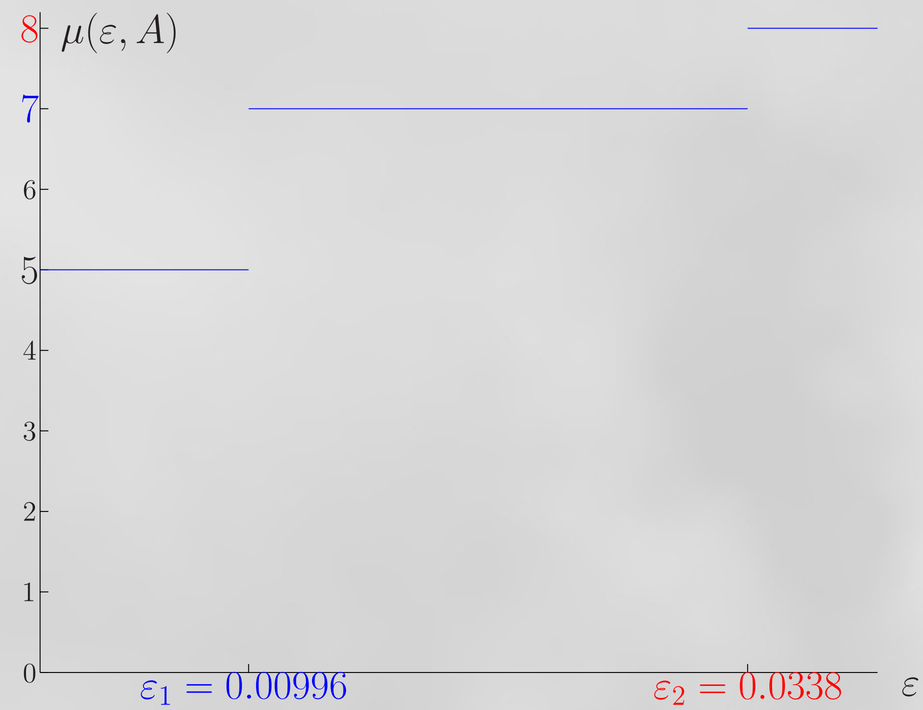
Figure 1

$$\mu(\varepsilon_1, A) = 7$$

$$\mu(\varepsilon_2, A) = 8$$




$\mu(\varepsilon_1, A) = 7$
$\mu(\varepsilon_2, A) = 8$
2
5
1
ε_1
ε_2


$\mu(\varepsilon, A)$
8
7
6
5
4
3
2
1
0
$\varepsilon_1 = 0.00996$
$\varepsilon_2 = 0.0338$
ε

The End

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