Smooth Jordanization

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Let $A : (a, b) \to \mathbb{C}^{n\times n}$ be a matrix function of class C^p defined on an open interval (a, b) of $\mathbb{R}$. Then for all $t \in (a, b)$, the matrix $A(t)$ is similar to a matrix J_t in Jordan form:

$$A(t) \approx J_t;$$

so, there exists an invertible matrix $P_t \in \mathbb{C}^{n\times n}$ such that

$$P_t^{-1} A(t)\, P_t = J_t.$$

In the following we try the solution of the

PROBLEM: *Can the function $t \mapsto P_t$ be chosen of class C^p?*

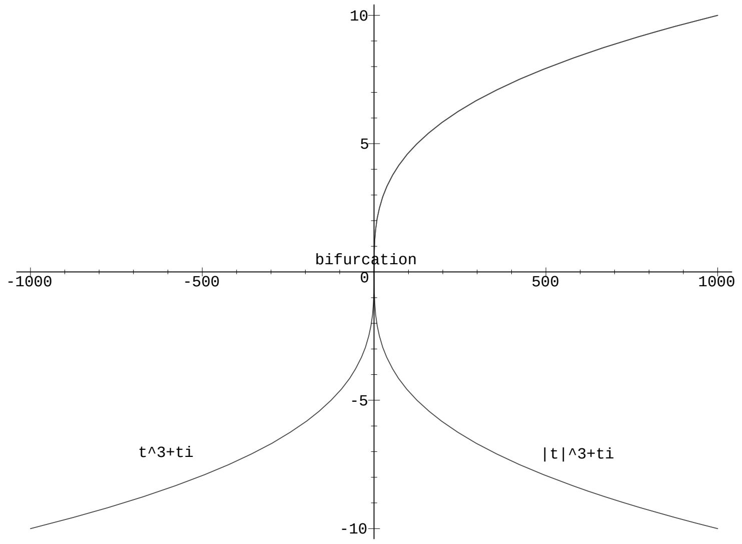
If such a function $P : (a, b) \to GL_n(\mathbb{C})$ exists then $J(t) := P(t)^{-1} A(t)\, P(t)$, $t \in (a, b)$, must be necessarily a function

$$J : (a, b) \to \mathbb{C}^{n\times n}$$

of class C^p.

Before giving a partial (but *generic*) answer to this question, we are going to put two examples.

Example 1.- Let $A_1(t) := \begin{pmatrix} t^3 + ti & 1 \\ 0 & |\, t\,|^3 + ti \end{pmatrix}$ for every $t \in \mathbb{R}$. The matrix function $A_1 : \mathbb{R} \to \mathbb{C}^{2\times 2}$ is of class C^2. The eigenvalues of $A_1(t)$ are $t^3 + ti$ and $-t^3 + ti$, distinct, if $t < 0$; when $t \geq 0$ $A_1(t)$ has only one eigenvalue $t^3 + ti$. The representative curves in the complex plane of $\alpha_1(t) := t^3 + ti$ and $\alpha_2(t) :=|\, t\,|^3 + ti$ are



The Jordan form of $A_1(t)$ is $J_t := \begin{pmatrix} t^3+ti & 0 \\ 0 & -t^3+ti \end{pmatrix}$ if $t < 0$, and $J_t := \begin{pmatrix} t^3+ti & 1 \\ 0 & t^3+ti \end{pmatrix}$ if $t \geq 0$. Therefore, given that the entry $(1,2)$ of J_t jumps from 0 to 1 when t pass through $t=0$ from left to right, there does not exist a function $P : \mathbb{R} \to GL_2(\mathbb{C})$ of class C^2, not even continuous at $t=0$, such that for all $t \in \mathbb{R}$,

$$P(t)^{-1}\, A_1(t)\, P(t) = J_t.$$

Example 2.- Let $A_2(t) = \begin{pmatrix} t^3+ti & 0 \\ 0 & \mid t \mid^3 + ti \end{pmatrix}$. The functions (of class C^2) $\alpha_1(t) := t^3 + ti$ and $\alpha_2(t) :=\mid t \mid^3 + ti$ are such that the spectrum, $\sigma(A_2(t))$, is equal to $\{\alpha_1(t), \alpha_2(t)\}$. When $t<0$, $\alpha_1(t) \neq \alpha_2(t)$; when $t \geq 0$, $\alpha_1(t) = \alpha_2(t)$. But, in this case the constant matrix function $P(t) := I_2$ for all $t \in \mathbb{R}$ is of class C^2 (even of class C^∞) and for all $t \in \mathbb{R}$,

$$I_2^{-1}\, A_2(t)\, I_2 = \begin{pmatrix} t^3 + ti & 0 \\ 0 & |\, t\,|^3 + ti \end{pmatrix} = \begin{pmatrix} \alpha_1(t) & 0 \\ 0 & \alpha_2(t) \end{pmatrix} =: J_t.$$

Definition 1.- In these examples $t_0 = 0$ is a *point of bifurcation* of the spectrum of A_1 and A_2. Let us precise this concept. Let $n(t) := \mathrm{Card}\,\{\mu \in \mathbb{C} : \det(A(t) - \mu I) = 0\}$ for every $t \in (a, b)$. We say that a point $t_0 \in (a, b)$ is a *point of bifurcation* of the spectrum of A if the function n is nonconstant in every neighbourhood of t_0.

Definition 2.- If $M \in \mathbb{C}^{n\times n}$ and λ_0 is eigenvalue of M, the *root subspace* of M corresponding to λ_0 is the subspace

$$\mathcal{R}_{\lambda_0}(M) := \mathrm{Ker}\,(M - \lambda_0 I)^n.$$

Lemma 1 *Let $A : (a, b) \to \mathbb{C}^{n\times n}$ be a matrix function of class C^p. Let $t_1 \in (a, b)$ be such that there exists a neighbourhood $V(t_1)$ of $t_1, V(t_1) \subset (a, b)$, satisfying that*

$$\mathrm{Card}\,\sigma(A(t))$$

is constant on $V(t_1)$, and equal to k. Then there exist k different functions $\alpha_i : V(t_1) \to \mathbb{C}$ of class C^p, $1 \le i \le k$, such that: (1) for all $t \in V(t_1)$,

$$\sigma(A(t)) = \{\alpha_1(t), \ldots, \alpha_k(t)\};$$

(2) moreover, for every $j \in \{1, \ldots, k\}$, there exists the limit (in the gap metric)

$$\lim_{t\to t_1} \mathcal{R}_{\alpha_j(t)}(A(t))$$

and it is equal to $\mathcal{R}_{\alpha_j(t_1)}(A(t_1))$.

Proof. By Lemma 4.2. of [1] there exist k different functions $\alpha_1, \ldots, \alpha_k : V(t_1) \to \mathbb{C}$ of class C^p such that for all $t \in V(t_1)$,

$$\sigma(A(t)) = \{\alpha_1(t), \ldots, \alpha_k(t)\}.$$

Let η be a positive real number such that the open balls $B(\alpha_j(t_1), \eta)$ $(1 \le j \le k)$ are pairwise disjoint. Then there exists a connected neighbourhood

$\mathcal{U}(t_1) \subset V(t_1)$ of t_1 such that for all $t \in \mathcal{U}(t_1) \quad \alpha_j(t) \in B(\alpha_j(t_1), \eta) \quad (1 \leq j \leq k)$.

As $A(t)$ has k distinct roots $\alpha_1(t), \ldots, \alpha_k(t)$ on $\mathcal{U}(t_1)$, $\mathcal{U}(t_1)$ is connected, and $\alpha_i (i = 1, \ldots, k)$ are continuous, by Lemma 2.1 of [2] it follows that the algebraic multiplicities of these roots $m_1, \ldots, m_k$ are constant on $\mathcal{U}(t_1)$.

By Proposition 2.2.5 of [3], it follows that

$$\dim \mathcal{R}_{\alpha_j(t)}(A(t)) = m_j$$

is constant on $\mathcal{U}(t_1) \quad (1 \leq j \leq k)$.

But $\mathcal{R}_{\alpha_j(t)}(A(t)) = \mathrm{Ker}\,[(A(t) - \alpha_j(t)\,I)^n]$ for every $t \in \mathcal{U}(t_1)$. Let us take a sequence $\tau_1, \tau_2, \ldots, \tau_q, \ldots \in \mathcal{U}(t_1)$ converging to t_1. By Theorem 15.1.3 of [3] we have that the limit

$$\lim_{q \to \infty} \mathrm{Ker}\,[(A(\tau_q) - \alpha_j(\tau_q) I)^n]$$

exists and it is equal to

$$\mathrm{Ker}\,[(A(t_1) - \alpha_j(t_1) I)^n].$$

As this is true for every sequence $\tau_1, \tau_2, \ldots, \tau_q, \ldots$ of points of $\mathcal{U}(t_1)$ converging to t_1, we deduce that there exists the limit

$$\lim_{t \to t_1} \mathcal{R}_{\alpha_j(t)}(A(t))$$

and it is equal to $\mathcal{R}_{\alpha_j(t_1)}(A(t_1)) \quad (1 \leq j \leq k)$.

□

Lemma 2 *Let $A : (a, b) \to \mathbb{C}^{n \times n}$ of class C^p, let*

$$r := \max_{t \in (a,b)} \ \mathrm{Card}\, \sigma(A(t)).$$

Let $\alpha_1, \ldots, \alpha_r : (a, b) \to \mathbb{C}$ be functions of class C^p such that for every $t \in (a, b)$,

$$\sigma(A(t)) = \{\alpha_1(t), \ldots, \alpha_r(t)\}.$$

Let $t_0 \in (a, b)$ be a point of bifurcation of the spectrum of A.

Then, there exist a number $\epsilon > 0$ and two subsets $\{j_1, \ldots, j_u\}$ and $\{k_1, \ldots, k_v\}$ of $\{1, \ldots, r\}$ such that for every $t \in (t_0 - \epsilon, t_0)$, the numbers $\alpha_{j_1}(t), \ldots, \alpha_{j_u}(t)$ are the distinct eigenvalues of $A(t)$, and for every $t \in (t_0, t_0 + \epsilon)$, the numbers $\alpha_{k_1}(t), \ldots, \alpha_{k_v}(t)$ are the distinct eigenvalues of $A(t)$.

Proof. It follows from results in [1].

$\square$

A partial solution to the Problem is here:

Theorem 3 *Let $A : (a, b) \to \mathbb{C}^{n\times n}$ be a matrix function of class C^p. Let*

$$r := \max_{t\in(a,b)} \operatorname{Card} \sigma(A(t)).$$

Suppose that the points of bifurcation of the spectrum of A, if there exist, are isolated.

Then, in order that there exists a function $P : (a, b) \to \mathbb{C}^{n\times n}$ of class C^p such that for every $t \in (a, b)$, the matrix $P(t)$ is invertible and the matrix $P(t)^{-1} A(t) P(t)$ is a Jordan matrix, it is necessary and sufficient that the two following conditions are satisfied:

Condition 1: *For every $k = 1, 2, \dots , n$, the number of $k \times k$ Jordan blocks of $A(t)$ is constant on (a, b).*

Condition 2: *There exist functions $\alpha_1, \dots , \alpha_r : (a, b) \to \mathbb{C}$ of class C^p such that:*

(2.1) *for all $t \in (a, b)$, $\sigma(A(t)) = \{\alpha_1(t), \dots , \alpha_r(t)\}$,*

(2.2) *for each point of bifurcation t_0 of the spectrum of A, let $\epsilon > 0$ and $\{j_1, \dots , j_u\}$, $\{k_1, \dots , k_v\}$ be the real number and the subsets of $\{1, \dots ,r\}$ associated to t_0 as in Lemma 2; then it must hold that*

$$\bigoplus_{i=1}^{u} \lim_{t\to t_0^-} \mathcal{R}_{\alpha_{j_i}(t)}(A(t)) = \mathbb{C}^n = \bigoplus_{i=1}^{v} \lim_{t\to t_0^+} \mathcal{R}_{\alpha_{k_i}(t)}(A(t)).$$

Proof. It follows the argumentation lines of [4].

$\square$

Revisited Examples

The number 0 is the only bifurcation point of the spectrum of A_1 and A_2.

If $t < 0$,

$$A_1(t) - \alpha_1(t)\, I_2 = \begin{pmatrix} 0 & 1 \\ 0 & -2t^3 \end{pmatrix};$$

$$A_1(t) - \alpha_2(t)\, I_2 = \begin{pmatrix} 2t^3 & 1 \\ 0 & 0 \end{pmatrix};$$

from here,

$$\mathcal{R}_{\alpha_1(t)}(A_1(t)) = < \begin{pmatrix} 1 \\ 0 \end{pmatrix} >; \;\; \mathcal{R}_{\alpha_2(t)}(A_1(t)) = < \begin{pmatrix} 1 \\ -2t^3 \end{pmatrix} > .$$

We deduce that

$$\lim_{t\to 0^-} \mathcal{R}_{\alpha_1(t)}(A_1(t)) = < \begin{pmatrix} 1 \\ 0 \end{pmatrix} >; \;\; \lim_{t\to 0^-} \mathcal{R}_{\alpha_2(t)}(A(t)) = < \begin{pmatrix} 1 \\ 0 \end{pmatrix} > .$$

But the sum $< \binom{1}{0} > + < \binom{1}{0} >$ is not direct and also $\neq \mathbb{C}^2$. So it is lacking the condition 2.2 of Theorem 1. It is not possible to reduce smoothly the matrix function A_1 to Jordan form.

On the other hand,
if $t < 0$,

$$A_2(t) - \alpha_1(t)\, I_2 = \begin{pmatrix} 0 & 0 \\ 0 & -2t^3 \end{pmatrix};$$

$$A_2(t) - \alpha_2(t)\, I_2 = \begin{pmatrix} 2t^3 & 0 \\ 0 & 0 \end{pmatrix};$$

we derive

$$\mathcal{R}_{\alpha_1(t)}(A_2(t)) = < \begin{pmatrix} 1 \\ 0 \end{pmatrix} >; \;\; \mathcal{R}_{\alpha_2(t)}(A_2(t)) < \begin{pmatrix} 0 \\ 1 \end{pmatrix} >$$

If $t > 0$,

$$A_2(t) - \alpha_1(t)\, I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

from here,

$$\mathcal{R}_{\alpha_1(t)}(A_2(t)) = \mathbb{C}^2 .$$

It is easy too see that

$$\lim_{t\to 0^-} \mathcal{R}_{\alpha_1(t)}(A_2(t)) \oplus \lim_{t\to 0^-} \mathcal{R}_{\alpha_2(t)}(A_2(t)) = \mathbb{C}^2 = \lim_{t\to 0^+} \mathcal{R}_{\alpha_1(t)}(A_2(t)).$$

So, this corroborates that the matrix function A_2 is smoothly reducible to Jordan form.

References

[1] J.C. Evard, On matrix functions which commute with their derivative, *Linear Algebra Appl.* **68** (1985) 145–178.

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[4] G.P.A. Thijsse, Global holomorphic similarity to a Jordan form, *Results in Mathematics*, **8**, n°. 1 (1985) 78–87.