Nearest derogatory matrix when varying into a submatrix*

José María González de Durana† Juan-Miguel Gracia‡

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Abstract

Let G be a nonderogatory complex matrix. We find a derogatory complex matrix that is one of the nearest to G among all possible matrices obtained by varying only the entries of a bottom right submatrix of G.

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1 Introduction

In this paper we use the following notation. By $\mathbb{C}$ we denote the field of complex numbers, and $\mathbb{C}^{m\times n}$ the set of $m \times n$ matrices with entries in $\mathbb{C}$. As usual, I_n or I denotes the identity matrix of order n; in particular $I_1 = (1)$. We always will use the spectral norm over $\mathbb{C}^{p\times q}$:

$$\|M\| = \max_{\substack{x \in \mathbb{C}^{q\times 1} \\ \|x\|_2 = 1}} \|Mx\|_2, \; M \in \mathbb{C}^{p\times q}$$

The singular values of a matrix M are denoted by $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_k(M)$, where $k = \min(p, q)$. It is well known that $\|M\| = \sigma_1(M)$. The Moore-Penrose inverse of M is denoted by $M^\dagger$ and M^* denotes the conjugate transpose of M. And, when $p = q$, we denote by $\sigma(M)$ the spectrum or set of distinct eigenvalues of M.

Let $A \in \mathbb{C}^{n\times n}$; the geometric multiplicity of an eigenvalue λ_0 of A is the number of Jordan blocks associated to λ_0 into the Jordan canonical form of A; we denote this number by $\mathrm{gm}(\lambda_0, A)$. So $\mathrm{gm}(\lambda_0, A)$ is the maximum number of linearly independent eigenvectors of A associated to λ_0; this implies that

$$\mathrm{gm}(\lambda_0, A) = \dim \mathrm{Ker}\,(\lambda_0 I_n - A).$$

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†Department of System Engineering and Automatics, The University of the Basque Country, EUITI, Nieves Cano,12 E-01006 Vitoria-Gasteiz, Spain. E-mail: jtpgogaj@vc.ehu.es

‡Department of Applied Mathematics and Statistics, The University of the Basque Country, Faculty of Pharmacy, P.O. Box 450, E-01080 Vitoria-Gasteiz, Spain.
E-mail: mepgrmej@vc.ehu.es

Definition 1 *A matrix $A \in \mathbb{C}^{n\times n}$ is called* nonderogatory *if its characteristic and minimal polynomials coincide; otherwise it is* derogatory *[9, p.240].*

It can be shown that each of the following conditions are equivalent:

(i) A is nonderogatory.

(ii) There is only one Jordan block associated to each eigenvalue of A.

(iii) For all $\lambda \in \mathbb{C}$ $\operatorname{rank}(\lambda I_n - A) \geq n-1$.

(iv) The matrices $I_n, A, A^2, \dots, A^{n-1}$ are linearly independent.

A complex number λ_0 is called a *derogatory eigenvalue* of a matrix $A \in \mathbb{C}^{n\times n}$ if $\operatorname{gm}(\lambda_0, A) \geq 2$. It is clear that a matrix is derogatory if and only if it has some derogatory eigenvalue.

We denote by $\mathcal{D} \subset \mathbb{C}^{n\times n}$ the set of complex derogatory $n\times n$ matrices and by $\mathcal{D}^c$, complement of $\mathcal{D}$ in $\mathbb{C}^{n\times n}$, the set of complex nonderogatory $n\times n$ matrices.

It is known that the set $\mathcal{D}^c \subset \mathbb{C}^{n\times n}$ of nonderogatory matrices is open in $\mathbb{C}^{n\times n}$ and that $\mathcal{D}$ is therefore a closed set. Then, given a matrix $\mathsf{D} \in \mathcal{D}^c$, if we consider a closed ball $\overline{B}(\mathsf{D}, \rho) \subset \mathbb{C}^{n\times n}$, with center at D and radius ρ, it makes sense to find the distance from D to the compact set $\mathcal{D} \cap \overline{B}(\mathsf{D}, \rho)$ of derogatory matrices in the ball.

The problem of finding

$$\min\{\|\mathsf{Y} - \mathsf{D}\| \ : \ \mathsf{Y} \text{ derogatory}\}$$

was addressed in [6, Corollary 4.2]. There its authors calculated this minimum value and also the matrix where it is attained. They obtained the formula

$$\min_{\substack{\mathsf{Y} \in \mathbb{C}^{n\times n} \\ \text{derogatory}}} \|\mathsf{Y} - \mathsf{D}\| = \min_{\lambda\in\mathbb{C}} \sigma_{n-1}(\lambda I_n - \mathsf{D}) \tag{1.1}$$

for the minimum and they also proved that if $\lambda_0 \in \mathbb{C}$ is a point where the function $\lambda \mapsto \sigma_{n-1}(\lambda I_n - \mathsf{D})$ attains its minimum value, then a matrix Y_1 where the minimum of the left-hand side of (1.1) is reached is given by

$$\mathsf{Y}_1 = \mathsf{D} + s_{n-1}u_{n-1}v_{n-1}^* + s_n u_n v_n^*$$

where

$$s_i, u_i, v_i, \ (i = n-1, n),$$

are the two last singular values and singular vectors of the matrix $\lambda_0 I_n - \mathsf{D}$. Moreover λ_0 is an eigenvalue of Y_1 with geometric multiplicity equal to 2.

The main result we obtain in this article (Theorem 4.1) generalizes this result to the case in which it is not allowed varying the whole matrix but only into a submatrix. Let G be an $n \times n$ complex nonderogatory matrix

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

partitioned into four blocks $A \in \mathbb{C}^{n_1\times n_1}$, $B \in \mathbb{C}^{n_1\times n_2}$, $C \in \mathbb{C}^{n_2\times n_1}$, $D \in \mathbb{C}^{n_2\times n_2}$.

We are going to find the distance from D to the set of matrices $Y \in \mathbb{C}^{n_2 \times n_2}$ such that the matrix

$$G_Y = \begin{pmatrix} A & B \\ C & Y \end{pmatrix}$$

is derogatory (in case of this set is not empty):

$$\min_{\substack{Y \in \mathbb{C}^{n_2 \times n_2} \\ G_Y \in \mathcal{D}}} \|Y - D\|. \tag{1.2}$$

Also we are going to find a matrix $Y_0 \in \mathbb{C}^{n_2 \times n_2}$ where this constrained minimum is attained.

In order to do that we will use some results from the papers [3], [12] which point out what are the possible ranks of all the matrices in the form

$$\mathsf{G}_\mathsf{X} := \begin{matrix} & \begin{matrix} n_1 & n_2 \end{matrix} & \\ & \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{X} \end{pmatrix} & \begin{matrix} m_1 \\ m_2 \end{matrix} \end{matrix},$$

by varying X in $\mathbb{C}^{m_2 \times n_2}$, and what is the nearest matrix, of this form, to the previously fixed matrix

$$\mathsf{G} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix}$$

and such that $\operatorname{rank} \mathsf{G}_\mathsf{X} < \operatorname{rank} \mathsf{G}$.

A summary of results from [3, Theorem 3] and [12, Theorem 2.1], which we need here, is the following theorem.

Theorem 1.1 *With the previous notations, let*

$$\mathsf{M} := (I - \mathsf{A}\mathsf{A}^\dagger)\mathsf{B}, \quad \mathsf{N} := \mathsf{C}(I - \mathsf{A}^\dagger\mathsf{A}).$$

and let

$$\rho := \operatorname{rank} \mathsf{A} + \operatorname{rank} \mathsf{M} + \operatorname{rank} \mathsf{N}$$

Then $\operatorname{rank} \mathsf{G}_\mathsf{X}$ *must satisfy*

$$\rho \leq \operatorname{rank} \mathsf{G}_\mathsf{X}$$

for all matrix $\mathsf{X} \in \mathbb{C}^{m_2 \times n_2}$.
Also it is true that there exists a matrix $\mathsf{Z} \in \mathbb{C}^{m_2 \times n_2}$ *such that*

$$\operatorname{rank} \mathsf{G}_\mathsf{Z} = \rho.$$

Moreover, for all $\mathsf{X} \in \mathbb{C}^{m_2 \times n_2}$,

$$\operatorname{rank} \mathsf{G}_\mathsf{X} = \rho + \operatorname{rank} S(\mathsf{X})$$

where

$$S(\mathsf{X}) := (I - \mathsf{N}\mathsf{N}^\dagger)(\mathsf{X} - \mathsf{C}\mathsf{A}^\dagger\mathsf{B})(I - \mathsf{M}^\dagger\mathsf{M})$$

If r is an integer which satisfy the inequalities

$$\rho \leq r < \operatorname{rank} \mathsf{G}$$

then a matrix $\mathsf{X}_0 \in \mathbb{C}^{m_2 \times n_2}$ *such that*

$$\|\mathsf{X}_0 - \mathsf{D}\| = \min\{\|\mathsf{X} - \mathsf{D}\| \ : \ \operatorname{rank} \mathsf{G}_\mathsf{X} \leq r\}$$

is given by the formula

$$\mathsf{X}_0 := \mathsf{D} - U \operatorname{diag}\Big(0, \ldots, 0, \sigma_{p+1}\big(S(\mathsf{D})\big), \sigma_{p+2}\big(S(\mathsf{D})\big), \ldots, \sigma_l\big(S(\mathsf{D})\big)\Big) V^* \quad (1.3)$$

in which:

(i) $p := r - \rho$

(ii) $U \in \mathbb{C}^{m_2 \times m_2}$, $V \in \mathbb{C}^{n_2 \times n_2}$ *are the unitary matrices which appear in the singular value decomposition of the matrix* $S(\mathsf{D})$*:*

$$U^* S(\mathsf{D}) V = \operatorname{diag}\Big(\sigma_1\big(S(\mathsf{D})\big), \ldots, \sigma_p\big(S(\mathsf{D})\big), \sigma_{p+1}\big(S(\mathsf{D})\big), \ldots, \sigma_l\big(S(\mathsf{D})\big)\Big) \in \mathbb{C}^{m_2 \times n_2}$$

(iii) $l := \min\{m_2, n_2\}$

(iv) $\operatorname{diag}\Big(\sigma_1\big(S(\mathsf{D})\big), \ldots, \sigma_p\big(S(\mathsf{D})\big), \sigma_{p+1}\big(S(\mathsf{D})\big), \ldots, \sigma_l\big(S(\mathsf{D})\big)\Big)$ *is the* $m_2 \times n_2$ *matrix* $\Delta = (d_{ij})$*, not necessarily squared, such that*

$$d_{ij} := \begin{cases} 0 & \text{if} \quad i \neq j \\ \sigma_i\big(S(\mathsf{D})\big) & \text{if} \quad i = j \end{cases}$$

(v) $\operatorname{diag}\Big(0, \ldots, 0, \sigma_{p+1}\big(S(\mathsf{D})\big), \sigma_{p+2}\big(S(\mathsf{D})\big), \ldots, \sigma_l\big(S(\mathsf{D})\big)\Big)$ *is the matrix from (iv) with the change* $d_{ii} = 0$ *for* $i = 1, \ldots, p$*.*

From (1.3) it results obvious that

$$\min\{\|\mathsf{X} - \mathsf{D}\| : \ \mathsf{X} \in \mathbb{C}^{m_2 \times n_2}, \ \operatorname{rank} \mathsf{G}_{\mathsf{X}} \leq r\} = \sigma_{p+1}\big(S(\mathsf{D})\big).$$

This paper is organized as follows: In Section 2 we address the question of existence of derogatory matrices in the shape $\left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right)$ with fixed A, B and C and variable Y; we will see that a such matrix exists always if the size of Y is greater than or equal to 2. Section 3 is devoted to an important real function h_{n-2} defined on a plane domain constituted by $\mathbb{R}^2$ minus some eigenvalues of A, if the size of D is greater than or equal to 2; when this size is equal to 1, the definition set of the function h_{n-2} is a subset of the spectrum of A (so, it is finite). Section 4 deals with the conversion of the constrained minimization problem (1.2) in a problem of global minimization of the function h_{n-2} on its domain. Finally, in Section 5 we consider the related question of finding where are the derogatory eigenvalues of all matrices $\left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right)$ with Y adequately close to D; this is linked with the concept of pseudospectrum [10, 11].

2 Existence of derogatory matrices with constraints

Before we consider the problem of finding the minimum of the set

$$\{\|Y - D\| : \ Y \in \mathbb{C}^{n_2 \times n_2}, \ \begin{pmatrix} A & B \\ C & Y \end{pmatrix} \text{ is derogatory}\}, \quad (2.1)$$

let us seeing if this set is not empty.

Example 2.1 If

$$G = \left(\begin{array}{cc|c} 0 & 1 & 2 \\ 0 & 0 & 3 \\ \hline 0 & 0 & 0 \end{array}\right) \tag{2.2}$$

with $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right)$, $D = \left(\begin{smallmatrix} 0 \end{smallmatrix}\right)$, it is obvious that G is a nonderogatory matrix because its minimal polynomial is λ^3. But for all $y \in \mathbb{C},\ y \neq 0$, the matrix

$$G_y = \left(\begin{array}{cc|c} 0 & 1 & 2 \\ 0 & 0 & 3 \\ \hline 0 & 0 & y \end{array}\right)$$

is similar to the nonderogatory matrix

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & y \end{array}\right)$$

Therefore, in this case there do not exist derogatory matrices in the form

$$\left(\begin{array}{cc|c} 0 & 1 & 2 \\ 0 & 0 & 3 \\ \hline 0 & 0 & y \end{array}\right)$$

Let us seeing a second example.

Example 2.2 Let

$$G = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 2 \end{array}\right) \tag{2.3}$$

with $A = \left(\begin{smallmatrix} 0 \end{smallmatrix}\right), B = \left(\begin{smallmatrix} 0 \end{smallmatrix}\right), C = \left(\begin{smallmatrix} 0 \end{smallmatrix}\right), D = \left(\begin{smallmatrix} 2 \end{smallmatrix}\right)$. The matrix G is nonderogatory, and for all value of $y \in \mathbb{C},\ y \neq 0$, also is nonderogatory the matrix

$$G_y = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & y \end{array}\right).$$

The only derogatory matrix in the form

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & y \end{array}\right)$$

is

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right)$$

So in this second example, the set (2.1) only has one element: the number 2.

In these two examples, $n_2 = 1$. Let us see that when $n_2 \geq 2$, the situation changes. The following proposition give us a sufficient condition so that the set in (2.1) is not empty.

Proposition 2.1 *Let n_1, n_2 be positive integers, let $A \in \mathbb{C}^{n_1 \times n_1}$, $B \in \mathbb{C}^{n_1 \times n_2}$, $C \in \mathbb{C}^{n_2 \times n_1}$ and let $n := n_1 + n_2$. If $n_2 \geq 2$ then there exist matrices $Y \in \mathbb{C}^{n_2 \times n_2}$ such that the matrix*

$$G_Y := \begin{pmatrix} A & B \\ C & Y \end{pmatrix} \in \mathbb{C}^{n \times n}$$

is derogatory.

PROOF: Let λ_0 be a complex number which is not an eigenvalue of A. Let

$$Y_0 := \lambda_0 I_{n_2} - C(\lambda_0 I_{n_1} - A)^{-1} B \tag{2.4}$$

We define

$$\begin{aligned} M(\lambda_0) &:= \ [I - (\lambda_0 I_{n_1} - A)(\lambda_0 I_{n_1} - A)^{-1}](-B), \\ N(\lambda_0) &:= \ (-C)[I - (\lambda_0 I_{n_1} - A)^{-1}(\lambda_0 I_{n_1} - A)]. \end{aligned}$$

Obviously, $M(\lambda_0) = 0$ and $N(\lambda_0) = 0$. Then, by virtue of Theorem 1.1,

$$\begin{aligned} \operatorname{rank}(\lambda_0 I_n - G_{Y_0}) &= \operatorname{rank}\left(\begin{array}{c|c} \lambda_0 I_{n_1} - A & -B \\ \hline -C & \lambda_0 I_{n_2} - Y_0 \end{array}\right) \\ &= \operatorname{rank}(\lambda_0 I_{n_1} - A) + \operatorname{rank} M(\lambda_0) + \operatorname{rank} N(\lambda_0) \\ &\quad + \operatorname{rank}(\lambda_0 I_{n_2} - Y_0 - C(\lambda_0 I_{n_1} - A)^{-1} B) \\ &= n_1 + \operatorname{rank}(\lambda_0 I_{n_2} - \lambda_0 I_{n_2} + C(\lambda_0 I_{n_1} - A)^{-1} B - C(\lambda_0 I_{n_1} - A)^{-1} B) \\ &= n_1 \end{aligned}$$

As $2 \leq n_2$, we have $n_1 \leq n_1 + n_2 - 2 = n - 2$. Therefore λ_0 is a derogatory eigenvalue of G_{Y_0} and this matrix is derogatory. □

Remark. Note this proposition proves even more: For each $\lambda \in \mathbb{C} \setminus \sigma(A)$ there exists a matrix $Y_\lambda \in \mathbb{C}^{n_2 \times n_2}$ such that λ is a derogatory eigenvalue of G_{Y_λ}.

Example 2.3 Let G be the matrix partitioned into four blocks

$$G = \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 \end{array}\right) = \left(\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right).$$

This matrix is nonderogatory because it has simple its five eigenvalues; they are approximately,

$$\sigma(G) = \{-1.80193, -1, -0.44504, 1.24698, 2\}$$

For all $y \in \mathbb{C} \setminus \sigma(G)$ we have that the matrix

$$G_y = \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & y \end{array}\right) = \left(\begin{array}{c|c} A & b \\ \hline c & y \end{array}\right) \tag{2.5}$$

is nonderogatory because of it has simple its eigenvalues. There exist only four derogatory matrices in form (2.5): they are the ones obtained when substituting y by any of the four eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of A. That is, for $i = 1, 2, 3, 4$,

$$\operatorname{rank}(\lambda_i I_5 - G_{\lambda_i}) = 3$$

and then λ_i is a derogatory eigenvalue of G_{λ_i}. So, the minimum

$$\min_{\substack{y \in \mathbb{C} \\ G_y \in \mathcal{D}}} |y-2| = \min_{1\le i\le 4} |\lambda_i - 2| \approx 2 - 1.24698 = 0.75302$$

Let us analyze what can happen to be, in general, when $n_2 = 1$. Let us suppose given $A \in \mathbb{C}^{(n-1)\times(n-1)}$, $b \in \mathbb{C}^{(n-1)\times 1}$, $c \in \mathbb{C}^{1\times(n-1)}$. For all $\lambda \in \mathbb{C}$ and for all $y \in \mathbb{C}$ let

$$\begin{aligned} M(\lambda) &:= [I_{n-1} - (\lambda I_{n-1} - A)(\lambda I_{n-1} - A)^\dagger](-b), \\ N(\lambda) &:= (-c)[I_{n-1} - (\lambda I_{n-1} - A)^\dagger(\lambda I_{n-1} - A)], \\ R(\lambda, y) &:= (I_1 - N(\lambda)N(\lambda)^\dagger)(\lambda - y - c(\lambda I_{n-1} - A)^\dagger b)(I_1 - M(\lambda)^\dagger M(\lambda)), \\ \rho(\lambda) &:= \operatorname{rank}(\lambda I_{n-1} - A) + \operatorname{rank} M(\lambda) + \operatorname{rank} N(\lambda) \end{aligned}$$

With these notations, by Theorem 1.1, we have

$$\operatorname{rank}\left(\lambda I_n - \begin{pmatrix} A & b \\ c & y \end{pmatrix}\right) = \operatorname{rank}\left(\begin{array}{c|c} \lambda I_{n-1} - A & -b \\ \hline -c & \lambda - y \end{array}\right) = \rho(\lambda) + \operatorname{rank} R(\lambda, y) \quad (2.6)$$

Moreover, if $\lambda \notin \sigma(A)$, then $\rho(\lambda) = n-1$ and by (2.6) it follows that for all $y \in \mathbb{C}$, λ is not a derogatory eigenvalue of the matrix

$$\begin{pmatrix} A & b \\ c & y \end{pmatrix}.$$

In the case of $\lambda \in \sigma(A)$ it must be $\rho(\lambda) \ge n-2$ because $\rho(\lambda) \le n-3$ is not possible since by (2.6)

$$\operatorname{rank}(\lambda I_n - G) = \rho(\lambda) + \operatorname{rank} R(\lambda, d) \ge n-1$$

and

$$0 \le \operatorname{rank} R(\lambda, d) \le 1;$$

so, there are two possibilities: either $\rho(\lambda) = n-2$ or $\rho(\lambda) \ge n-1$. If $\rho(\lambda) = n-2$ and y satisfies that

$$R(\lambda, y) = 0,$$

then λ is an derogatory eigenvalue of

$$\begin{pmatrix} A & b \\ c & y \end{pmatrix}.$$

If $\rho(\lambda) \ge n-1$, then for all $y \in \mathbb{C}$, λ is not a derogatory eigenvalue of

$$\begin{pmatrix} A & b \\ c & y \end{pmatrix}.$$

By summarizing all of preceding statements, we obtain the following proposition.

Proposition 2.2 *With the previous notations, for any $\lambda \in \mathbb{C}$ we define the set*

$$\mathcal{N}_\lambda := \{y \in \mathbb{C} : \ \lambda \textit{ is a derogatory eigenvalue of } \begin{pmatrix} A & b \\ c & y \end{pmatrix}\}$$

Then,

(i) If $\lambda \notin \sigma(A)$, we have $\mathcal{N}_\lambda = \emptyset$.

(ii) If $\lambda \in \sigma(A)$, it follows $\rho(\lambda) \geq n-2$.

(ii.1) In the case of $\rho(\lambda) = n-2$,

$$\mathcal{N}_\lambda := \{y \in \mathbb{C} : \quad R(\lambda, y) = 0\}.$$

If $N(\lambda)N(\lambda)^\dagger = (1)$ or $M(\lambda)^\dagger M(\lambda) = (1)$, then $\mathcal{N}_\lambda = \mathbb{C}$; in the opposite case, $\lambda - c(\lambda I_{n_1} - A)^\dagger b$ is the only element of $\mathcal{N}_\lambda$.

(ii.2) In the case of $\rho(\lambda) \geq n-1$, we have $\mathcal{N}_\lambda = \emptyset$.

Returning to Example 2.3, in the light of this proposition, we see that for $i = 1, 2, 3, 4$,

$$\rho(\lambda_i) = 3 = 5 - 2$$

and as $M(\lambda_i) = 0, N(\lambda_i) = 0, b = 0$ and $c = 0$ we have that

$$\mathcal{N}_{\lambda_i} = \{\lambda_i\},$$

λ_i is the only solution of $R(\lambda_i, y) = 0$, $y \in \mathbb{C}$.

Let us come back to the general case. Let n_1, n_2 be any positive integers, let $n = n_1 + n_2$ and let $A \in \mathbb{C}^{n_1 \times n_1}$, $B \in \mathbb{C}^{n_1 \times n_2}$, $C \in \mathbb{C}^{n_2 \times n_1}$. For each $\lambda \in \mathbb{C}$, let us denote by $\mathcal{N}_\lambda$ the set of matrices $Y \in \mathbb{C}^{n_2 \times n_2}$ such that λ is a derogatory eigenvalue of the $n \times n$ matrix

$$\begin{pmatrix} A & B \\ C & Y \end{pmatrix},$$

or, with symbols,

$$\mathcal{N}_\lambda := \{Y \in \mathbb{C}^{n_2 \times n_2} : \operatorname{rank}(\lambda I_n - \left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right)) \leq n - 2\}$$

For some values of λ the set $\mathcal{N}_\lambda$ could be empty. Identifying a complex number y with the 1×1 matrix (y), we have analyzed some cases in Proposition 2.2. Let us see another example.

Example 2.4 Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -3 \\ 0 & 3 \end{pmatrix}$$

Let us define

$$\begin{aligned} M &:= -(I_2 - (I_2 - A)(I_2 - A)^\dagger)B, \\ N &:= -C(I_2 - (I_2 - A)^\dagger (I_2 - A)) \end{aligned}$$

Then, by Theorem 1.1, for all matrix $Y \in \mathbb{C}^{2 \times 2}$ we have that

$$\operatorname{rank}\left(1I_4 - \left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right)\right) \geq \operatorname{rank}(1I_2 - A)) + \operatorname{rank} M + \operatorname{rank} N = 3$$

Therefore, $\mathcal{N}_1 = \emptyset$.

Let, in general, $\Omega := \{\lambda \in \mathbb{C} : \mathcal{N}_\lambda \neq \emptyset\}$. Denoting by $\mathcal{N}$ the set of matrices $Y \in \mathbb{C}^{n_2 \times n_2}$ such that the matrix

$$\begin{pmatrix} A & B \\ C & Y \end{pmatrix}$$

is derogatory, it follows that

$$\mathcal{N} = \bigcup_{\lambda \in \Omega} \mathcal{N}_\lambda. \tag{2.7}$$

By (2.4) one can see that if $n_2 \geq 2$ then

$$\mathbb{C} \setminus \sigma(A) \subset \Omega, \tag{2.8}$$

but Ω can contain eigenvalues of A as we have seen. This is a very significant question in the sequel.

By Proposition 2.2, in the case of $n_2 = 1$,

$$\Omega := \{\lambda \in \sigma(A) : \rho(\lambda) = n - 2\}. \tag{2.9}$$

Example 2.5 Consider the matrix

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} := \left(\begin{array}{c|cc} 0 & 1 & 2 \\ \hline 0 & 0 & 3 \\ 0 & 0 & 0 \end{array}\right).$$

We define the matrix

$$G_Y = \begin{pmatrix} A & B \\ C & Y \end{pmatrix} := \left(\begin{array}{c|cc} 0 & 1 & 2 \\ \hline 0 & x & y \\ 0 & z & u \end{array}\right), \tag{2.10}$$

and we are going to compute explicitly the minimum of the set

$$\{\|Y - D\| : Y \in \mathbb{C}^{n_2 \times n_2}, \ \left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right) \text{ is derogatory}\}.$$

By Definition 1 we know that the matrix G_Y is derogatory if and only if the matrices I_3, G_Y, and G_Y^2 are linearly dependent. Therefore the set of derogatory matrices G_Y is characterized for all complex numbers x, y, z, u that verify the equality

$$\beta I_3 + \alpha G_Y = G_Y^2$$

for some $\alpha, \beta \in \mathbb{C}$, not both zero. Then we have

$$\begin{pmatrix} \beta & \alpha & 2\alpha \\ 0 & \beta + \alpha x & \alpha y \\ 0 & \alpha z & \beta + \alpha u \end{pmatrix} = \begin{pmatrix} 0 & x + 2z & y + 2u \\ 0 & x^2 + yz & xy + yu \\ 0 & xz + zu & yz + u^2 \end{pmatrix}$$

It is obvious that β must be zero and also

$$x + 2z = \alpha = \frac{1}{2}(y + 2u)$$
$$yz + u^2 = \alpha u = \frac{1}{2}(y + 2u)$$

which, for $y \neq 0$, implies the relations

$$y = 2x, \quad u = 2z$$

among the variables. So the matrices in the form

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & x & 2x \\ 0 & z & 2z \end{pmatrix},$$

for all $x, z \in \mathbb{C}$, $x \neq 0$, are the only derogatory matrices of shape (2.10) sufficiently close to G.

Then the value

$$\min_{G_Y \text{ derogatory}} \|G_Y - G\| = \min_{G_Y \text{ derogatory}} \|Y - D\|$$

is equal to the value

$$\min_{(x,z)\in\mathbb{C}^2} \left\| \begin{pmatrix} 0 & 1 & 2 \\ 0 & x & 2x \\ 0 & z & 2z \end{pmatrix} - \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \right\| = \min_{(x,z)\in\mathbb{C}^2} \left\| \begin{pmatrix} x & 2x-3 \\ z & 2z \end{pmatrix} \right\|$$

and it can be explicitly computed by global minimization of a real function of four real variables, without constraints. As we are using the 2-norm, we have $\|Y - D\| = \sigma_1(Y - D)$ and its value, computed with Maple is

$$\begin{aligned} \sigma_1(Y-D) = & \frac{1}{2}\big(10\,x\,\overline{x} - 12\,x + 18 - 12\,\overline{x} + 10\,z\,\overline{z} \\ & +2(81 - 108\,x - 60\,\overline{x}\,z\,\overline{z} + 54\,z\,\overline{z} - 60\,x\,z\,\overline{z} + 162\,x\,\overline{x} - 108\,\overline{x} + 25\,x^2\,\overline{x}^2 \\ & - 60\,x^2\,\overline{x} - 60\,x\,\overline{x}^2 + 36\,x^2 + 36\,\overline{x}^2 + 25\,z^2\,\overline{z}^2 + 50\,x\,\overline{x}\,z\,\overline{z})^{1/2}\big)^{1/2} \end{aligned}$$

The minimum value of this function, computed with the command `minimize` of Maple, is

$$\frac{3}{\sqrt{5}}.$$

Later, in Example 4.1, we will calculate this value once more using the function $h_{n-2}(\lambda)$, to be defined in (3.8) , as Theorem 4.1 will show.

3 An important real function of a complex variable

Let

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{(n_1+n_2)\times(n_1+n_2)},$$

be an $n \times n$, four block partitioned, nonderogatory matrix. In order to use the Theorem 1.1, for each $\lambda \in \mathbb{C}$ we define the matrices

$$\begin{aligned} M(\lambda) := & \quad [I_{n_1} - (\lambda I_{n_1} - A)(\lambda I_{n_1} - A)^\dagger](-B), \\ N(\lambda) := & \quad (-C)[I_{n_1} - (\lambda I_{n_1} - A)^\dagger(\lambda I_{n_1} - A)]. \end{aligned}$$

It is easy to see that the set $\Omega := \{\lambda \in \mathbb{C} : \ \mathcal{N}_\lambda \neq \emptyset\}$ before defined is equal to the set

$$\{\lambda \in \mathbb{C} : \ \operatorname{rank}(\lambda I_{n_1} - A) + \operatorname{rank} M(\lambda) + \operatorname{rank} N(\lambda) \leq n-2\}. \tag{3.1}$$

For all $\lambda \in \Omega$ we define

$$p(\lambda) := n - 2 - \big(\operatorname{rank}(\lambda I_{n_1} - A) + \operatorname{rank} M(\lambda) + \operatorname{rank} N(\lambda)\big). \tag{3.2}$$

By (3.1) it follows that

$$p(\lambda) \geq 0. \tag{3.3}$$

Proposition 3.1 *For all $\lambda \in \Omega$, $\quad p(\lambda) \leq n_2 - 1$.*

Proof: For all $\lambda \in \mathbb{C}$ we have

$$\operatorname{rank}\left(\begin{array}{c|c} \lambda_0 I_{n_1} - A & -B \\ \hline -C & \lambda_0 I_{n_2} - D \end{array}\right) \geq n - 1. \tag{3.4}$$

By calling

$$\rho_1(\lambda) := \operatorname{rank}(\lambda I_{n_1} - A) + \operatorname{rank} M(\lambda) + \operatorname{rank} N(\lambda) \tag{3.5}$$

we have

$$\operatorname{rank}\left(\begin{array}{c|c} \lambda_0 I_{n_1} - A & -B \\ \hline -C & \lambda_0 I_{n_2} - D \end{array}\right) = \rho_1(\lambda) + \operatorname{rank} D_1(\lambda), \tag{3.6}$$

where

$$D_1(\lambda) := \big(I_{n_2} - N(\lambda)N(\lambda)^\dagger\big) \cdot \big(\lambda I_{n_2} - D - C(\lambda I_{n_1} - A)^\dagger B\big) \cdot \big(I_{n_2} - M(\lambda)^\dagger M(\lambda)\big) \tag{3.7}$$

is an $n_2 \times n_2$ matrix. ¿From (3.4) and (3.6) we deduce

$$n_1 + n_2 - 1 \leq \rho_1(\lambda) + \operatorname{rank} D_1(\lambda);$$

as $\operatorname{rank} D_1(\lambda) \leq n_2$, we have

$$n_1 + n_2 - 1 \leq \rho_1(\lambda) + n_2.$$

This implies that

$$n_1 - 1 \leq \rho_1(\lambda);$$

therefore

$$p(\lambda) = n_1 + n_2 - 2 - \rho_1(\lambda) \leq n_1 + n_2 - 2 - (n_1 - 1) = n_2 - 1$$

□

Let

$$\begin{array}{rccc} h_{n-2}: & \Omega & \rightarrow & \mathbb{R} \\ & \lambda & \mapsto & \sigma_{p(\lambda)+1}\big(D_1(\lambda)\big) \end{array} \tag{3.8}$$

be the function that associates to each complex number $\lambda \in \Omega$ the $(p(\lambda)+1)$-th singular value of the $n_2 \times n_2$ matrix $D_1(\lambda)$.

Let us assume now that $n_2 \geq 2$ which is the most interesting case. Theorem 3.3 summarizes some properties of the function h_{n-2}. Before of giving its statement, we need some previous results.

Lemma 3.2 *Let M_1, M_2, M_3 be $n \times n$ complex matrices. Then the following inequalities concerning their singular values are true:*

(i) $\sigma_n(M_1)\,\sigma_{n-1}(M_2)\,\sigma_n(M_3) \leq \sigma_{n-1}(M_1M_2M_3)$,

(ii) $\sigma_{n-1}(M_1M_2M_3) \leq \|M_1\|\,\|M_3\|\,\sigma_{n-1}(M_2)$.

Proof: ¿From a careful inspection of the statement of Theorem 4.10, p. 50, of [1] it follows inequality (i). From Theorem 4.12, p. 53, of [1] or Problem III.6.5, p. 75, of [2], it follows inequality (ii). □

Now let

$$F(\lambda) := \lambda I_{n_2} - D - C(\lambda I_{n_1} - A)^{-1}B;$$

here, $\lambda I_{n_2} - D$ is a polynomial matrix in the variable λ and $C(\lambda I_{n_1} - A)^{-1}B$ is a strictly proper rational matrix function because

$$\lim_{|\lambda|\to\infty} C(\lambda I_{n_1} - A)^{-1}B = 0.$$

Moreover, for each

$$\lambda \in \mathbb{C} \setminus \left(\sigma(A) \cup \sigma\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)\right),$$

we have

$$n = \operatorname{rank}\left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & \lambda I_{n_2} - D \end{array}\right) =$$

$$\operatorname{rank}(\lambda I_{n_1} - A) + \operatorname{rank} F(\lambda) = n_1 + \operatorname{rank} F(\lambda),$$

in virtue of formula (7), p. 46, of [9] on the Schur complement of $\lambda I_{n_1} - A$ in

$$\left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & \lambda I_{n_2} - D \end{array}\right).$$

Hence,

$$\operatorname{rank} F(\lambda) = n_2$$

and so $\det F(\lambda) \not\equiv 0$. Therefore, we can consider the *local Smith form* of the rational matrix function $F(\lambda)$ at λ_0, the complex number λ_0 being an eigenvalue of A:

$$F(\lambda) = E_1(\lambda)\,\operatorname{diag}[(\lambda - \lambda_0)^{\nu_1}, \ldots, (\lambda - \lambda_0)^{\nu_{n_2}}]\,E_2(\lambda), \tag{3.9}$$

where $E_1(\lambda)$ and $E_2(\lambda)$ are rational matrix functions that are defined and invertible at λ_0, and $\nu_1, \ldots, \nu_{n_2}$ are integers; these integers are uniquely determined by $F(\lambda)$ and λ_0 up to permutation and do not depend on the particular choice of the local Smith form (3.9); they are called the *partial multiplicities* of $F(\lambda)$ at λ_0. See Section 7.2, p. 218–219, of [5].

Theorem 3.3 *With the previous notations, let us assume $n_2 \geq 2$. Let $h_{n-2} : \Omega \to \mathbb{R}$ be the function we have defined in* (3.8)*. Then*

(i) the function h_{n-2} is continuous on $\Omega \setminus \sigma(A)$,

(ii) if $\lambda_0 \in \sigma(A)$ and the number of negative partial multiplicities of $F(\lambda)$ at λ_0 is greater than or equal to $n_2 - 1$, then

$$\lim_{\lambda \to \lambda_0} h_{n-2}(\lambda) = \infty,$$

(iii) if $\lambda_0 \in \sigma(A)$ and the number of negative partial multiplicities of $F(\lambda)$ at λ_0 is less than $n_2 - 1$, then there exists the limit

$$\lim_{\lambda \to \lambda_0} h_{n-2}(\lambda),$$

and it is a real number.

(iv)

$$\lim_{|\lambda| \to \infty} h_{n-2}(\lambda) = \infty.$$

PROOF:

(i) If $\lambda \in \Omega \setminus \sigma(A)$, then

$$(\lambda I_{n_1} - A)^{\dagger} = (\lambda I_{n_1} - A)^{-1}$$

and therefore

$$M(\lambda) = 0, \quad N(\lambda) = 0$$

and from (3.2) it follows

$$p(\lambda) = n_1 + n_2 - 2 - n_1 = n_2 - 2; \tag{3.10}$$

so that

$$h_{n-2}(\lambda) = \sigma_{n_2 - 1}(\lambda I_{n_2} - D - C(\lambda I_{n_1} - A)^{-1} B). \tag{3.11}$$

By virtue of the continuity of the function

$$\lambda \mapsto (\lambda I_{n_1} - A)^{-1}$$

on $\mathbb{C} \setminus \sigma(A)$ and because of being the singular values of a matrix continuous functions of it, it follows that the function

$$\lambda \mapsto \sigma_{n_2 - 1}(\lambda I_{n_2} - D - C(\lambda I_{n_1} - A)^{-1} B)$$

is continuous at each point $\lambda \in \Omega \setminus \sigma(A)$.

(ii) Call $\Delta(\lambda)$ the diagonal matrix

$$\mathrm{diag}[(\lambda - \lambda_0)^{\nu_1}, \ldots, (\lambda - \lambda_0)^{\nu_{n_2}}]$$

that appears in (3.9). Applying inequality (i) of Lemma 3.2 to product (3.9) we have

$$\sigma_{n_2}\big(E_1(\lambda)\big)\, \sigma_{n_2 - 1}\big(\Delta(\lambda)\big)\, \sigma_{n_2}\big(E_2(\lambda)\big) \leq \sigma_{n_2 - 1}\big(F(\lambda)\big). \tag{3.12}$$

It is easy to see that the singular values of $\Delta(\lambda)$ are

$$|\lambda - \lambda_0|^{\nu_1}, \ldots, |\lambda - \lambda_0|^{\nu_{n_2}},$$

(not necessarily ordered from largest to smallest). By the hypothesis on the negative partial multiplicities of $F(\lambda)$ at λ_0, we have that the (n_2-1)-th singular value of $\Delta(\lambda)$ (when ordered in nonincreasing order) is in the shape

$$\frac{1}{|\lambda-\lambda_0|^p},$$

with a positive integer p (the number p does not depend on λ!) for all λ *sufficiently* closed to λ_0 and different from it. Hence,

$$\lim_{\lambda\to\lambda_0} \sigma_{n_2-1}\big(\Delta(\lambda)\big) = \lim_{\lambda\to\lambda_0} \frac{1}{|\lambda-\lambda_0|^p} = \infty. \tag{3.13}$$

As $E_1(\lambda_0)$ and $E_2(\lambda_0)$ are invertible it follows

$$\lim_{\lambda\to\lambda_0} \sigma_{n_2}\big(E_1(\lambda)\big) = \sigma_{n_2}\big(E_1(\lambda_0)\big) > 0,$$

$$\lim_{\lambda\to\lambda_0} \sigma_{n_2}\big(E_2(\lambda)\big) = \sigma_{n_2}\big(E_2(\lambda_0)\big) > 0.$$

Therefore, by (3.13) we have

$$\lim_{\lambda\to\lambda_0} \sigma_{n_2}\big(E_1(\lambda)\big)\, \sigma_{n_2-1}\big(\Delta(\lambda)\big)\, \sigma_{n_2}\big(E_2(\lambda)\big) = \infty;$$

from here and (3.12) it follows

$$\lim_{\lambda\to\lambda_0} \sigma_{n_2-1}\big(F(\lambda)\big) = \infty.$$

(iii) Let q be the number of negative partial multiplicities of $F(\lambda)$ at λ_0. So, $q < n_2 - 1$. Permuting the elements of the diagonal of $\Delta(\lambda)$, if necessary, we can suppose that

$$\nu_1 < 0, \dots, \nu_q < 0, \nu_{q+1} \geq 0, \dots, \nu_{n_2} \geq 0.$$

Then the singular values of $\Delta(\lambda)$ are

$$\frac{1}{|\lambda-\lambda_0|^{-\nu_1}}, \dots, \frac{1}{|\lambda-\lambda_0|^{-\nu_q}}, |\lambda-\lambda_0|^{\nu_{q+1}}, \dots, |\lambda-\lambda_0|^{\nu_{n_2}}. \tag{3.14}$$

In the case of λ is sufficiently close to λ_0, the numbers

$$\frac{1}{|\lambda-\lambda_0|^{-\nu_1}}, \dots, \frac{1}{|\lambda-\lambda_0|^{-\nu_q}},$$

are the q greatest numbers in the list (3.14); thus,

$$\sigma_{n_2-1}\big(\Delta(\lambda)\big) = |\lambda-\lambda_0|^{\ell} \tag{3.15}$$

with ℓ an integer ≥ 0 (the number ℓ does not depend on λ!).

Taking into account (3.9), (3.15) and inequality (ii) in Lemma 3.2,

$$\begin{aligned} \sigma_{n_2-1}\big(F(\lambda)\big) &\leq \|E_1(\lambda)\|\, \|E_2(\lambda)\|\, \sigma_{n_2-1}\big(\Delta(\lambda)\big) \\ &= \|E_1(\lambda)\|\, \|E_2(\lambda)\|\, |\lambda-\lambda_0|^{\ell}. \end{aligned} \tag{3.16}$$

Given that $E_1(\lambda_0)$ and $E_2(\lambda_0)$ are invertible, $\|E_1(\lambda_0)\| > 0, \|E_2(\lambda_0)\| > 0$; then by (3.16) there exist a real number $M > 0$ and a deleted neighbourhood $\mathcal{N}$ of λ_0 such that for all $\lambda \in \mathcal{N}$, we have

$$\sigma_{n_2-1}\big(F(\lambda)\big) \leq M.$$

From this upper bound and due to the fact that $\sigma_{n_2-1}\big(F(\lambda)\big)$ is an algebraic function, it follows that there exists the limit

$$\lim_{\lambda \to \lambda_0} \sigma_{n_2-1}\big(F(\lambda)\big).$$

(iv) For all $\lambda \in \mathbb{C} \setminus \sigma(A)$, by [8, p.178, Theorem 3.3.16 (c)] we have

$$\begin{aligned} \big|\sigma_{n_2-1}\big(\lambda I_{n_2} - D - C(\lambda I_{n_1} - A)^{-1}B\big) - \sigma_{n_2-1}\big(\lambda I_{n_2} - D\big)\big| \\ \leq \| - C(\lambda I_{n_1} - A)^{-1}B\|. \end{aligned} \tag{3.17}$$

As $(\lambda I_{n_1} - A)^{-1}$ is a matrix of strictly proper rational functions in λ, we have

$$\| - C(\lambda I_{n_1} - A)^{-1}B\| \to 0 \tag{3.18}$$

when $|\lambda| \to \infty$. Given that $\sigma_{n_2-1}(\lambda I_{n_2} - D) \to \infty$ when $|\lambda| \to \infty$ [6, proof of Theorem 4.1], it follows from (3.17) and (3.18) that

$$\lim_{|\lambda| \to \infty} \sigma_{n_2-1}\big(\lambda I_{n_2} - D - C(\lambda I_{n_1} - A)^{-1}B\big) = \infty.$$

□

¿From this theorem it follows that there exists the minimum

$$\min_{\lambda \in \Omega} h_{n-2}(\lambda). \tag{3.19}$$

If $n_2 = 1$, by (2.9) the set Ω is finite; from which the minimum (3.19) exists for whatever value of n_2.

Example 3.1 Let us consider the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 1 & 2 & 3 \\ 0 & 2 & 4 & 5 \\ \hline 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

Here $n - 2 = 4 - 2 = 2$, $\Omega = \mathbb{C}(\equiv \mathbb{R}^2)$ and the graphic of the function

$$f(x, y) := h_2(x + yi),$$

defined in all $\mathbb{R}^2$, has been depicted in Figure 1.

The rational matrix function $F(\lambda)$ has no negative partial multiplicity at 1 and at 2. In this case there exist the double limits

$$\lim_{(x,y) \to (1,0)} f(x, y) \qquad \text{and} \qquad \lim_{(x,y) \to (2,0)} f(x, y),$$

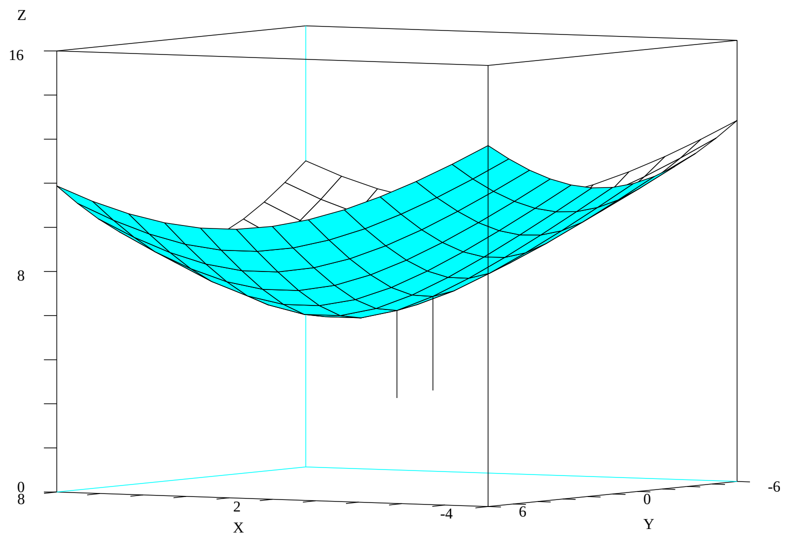

Figure 1: Graphical representation of the function $f(x, y)$

f is discontinuous at $(1, 0)$ and $(2, 0)$ (both belonging to the spectrum of A) and

$$\min_{(x,y)\in\mathbb{R}^2} f(x, y) = f(2, 0) \approx 3.2196$$

In addition to this, $f(1, 0) \approx 3.5355$, the minimum of f on $\Omega \setminus \{(1, 0), (2, 0)\}$ exists and

$$\min_{(x,y)\in\Omega\setminus\{(1,0),(2,0)\}} f(x, y) \approx 6.0414;$$

what is the meaning of this value? We will answer this question in Section 5 later.

The above example is a very special case because its block C is zero. Let us see what happens when this block is nonzero.

Example 3.2 Now we take the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 1 & 2 & 3 \\ 0 & 2 & 4 & 5 \\ \hline 13 & -2 & 3 & 6 \\ 0 & 7 & 0 & 4 \end{array}\right)$$

in which the block C of the previous example has been changed by $\left(\begin{smallmatrix} 13 & -2 \\ 0 & 7 \end{smallmatrix}\right)$. Figure 2 represents the function $h_{n-2}(\lambda)$ for this matrix. As you can see, the function tends to infinity when λ tends to the eigenvalues 1 and 2 of the block A. The rational matrix function $F(\lambda)$ has one negative partial multiplicity at its poles 1 and 2.

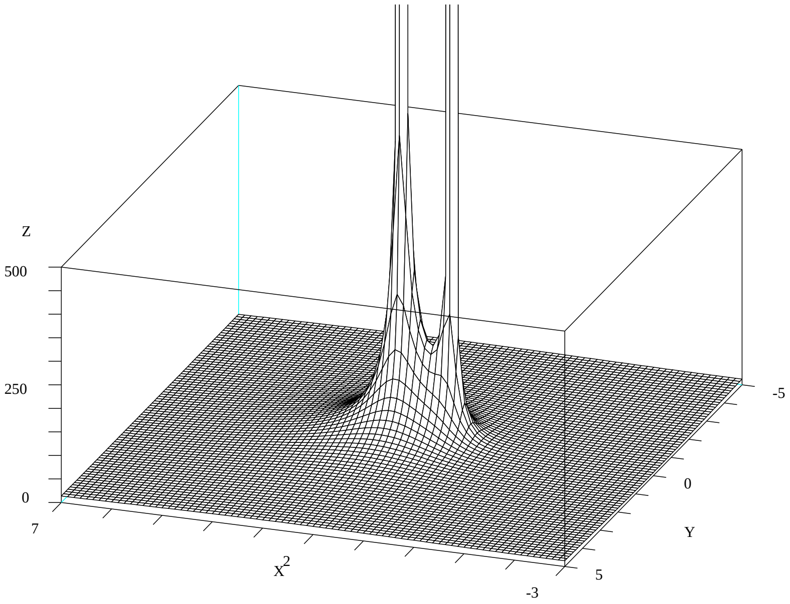

Figure 2: Graphical representation of the function $h_{n-2}(\lambda)$

4 Distance to the set of "derogatizing" matrices

Let the nonderogatory $n \times n$ complex matrix

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be partitioned into four blocks $A \in \mathbb{C}^{n_1 \times n_1}$, $B \in \mathbb{C}^{n_1 \times n_2}$, $C \in \mathbb{C}^{n_2 \times n_1}$, $D \in \mathbb{C}^{n_2 \times n_2}$ and let $G_Y := \left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right)$. Let $\mathcal{D} \subset \mathbb{C}^{n \times n}$ be the set of derogatory $n \times n$ complex matrices. In this section we give a solution for the problem of finding the minimum of the set

$$\{\|Y - D\| : \; Y \in \mathbb{C}^{n_2 \times n_2}, \quad G_Y \in \mathcal{D}\} \tag{4.1}$$

by means of the following theorem.

Theorem 4.1 *Using the preceding notation, let $A \in \mathbb{C}^{n_1 \times n_1}$, $B \in \mathbb{C}^{n_1 \times n_2}$, $C \in \mathbb{C}^{n_2 \times n_1}$, $D \in \mathbb{C}^{n_2 \times n_2}$ be matrices such that the $n \times n$ matrix*

$$G := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is nonderogatory. For each $\lambda \in \mathbb{C}$, let

$$\begin{aligned} M(\lambda) :=& \quad -[I_{n_1} - (\lambda I_{n_1} - A)(\lambda I_{n_1} - A)^\dagger]B, \\ N(\lambda) :=& \quad -C[I_{n_1} - (\lambda I_{n_1} - A)^\dagger(\lambda I_{n_1} - A)], \end{aligned}$$

and let Ω be the set

$$\{\lambda \in \mathbb{C} : \; \operatorname{rank}(\lambda I_{n_1} - A) + \operatorname{rank} M(\lambda) + \operatorname{rank} N(\lambda) \leq n - 2\}.$$

For each $\lambda \in \Omega$ we define:

(i) $p(\lambda) := n - 2 - \big(\operatorname{rank}(\lambda I_{n_1} - A) + \operatorname{rank} M(\lambda) + \operatorname{rank} N(\lambda)\big)$,

(ii) $D_1(\lambda) := \big(I_{n_2} - N(\lambda)N(\lambda)^\dagger\big) \cdot \big(\lambda I_{n_2} - D - C(\lambda I_{n_1} - A)^\dagger B\big) \cdot \big(I_{n_2} - M(\lambda)^\dagger M(\lambda)\big)$,

(iii) $h_{n-2}(\lambda) := \sigma_{p(\lambda)+1}\big(D_1(\lambda)\big)$.

Then

$$\min_{\substack{Y \in \mathbb{C}^{n_2\times n_2} \\ G_Y \in \mathcal{D}}} \|Y - D\| = \min_{\lambda\in\Omega} h_{n-2}(\lambda). \tag{4.2}$$

Moreover, if λ_0 is a complex number where the function $h_{n-2} : \Omega \to \mathbb{R}$ attains its minimum value, then a matrix Y_1 which minimizes the left-hand side of (4.2) is given by

$$Y_1 := D + U \operatorname{diag}(0, \ldots, 0, \tau_{p(\lambda_0)+1}, \ldots, \tau_{n_2})V^*, \tag{4.3}$$

where $U, V \in \mathbb{C}^{n_2\times n_2}$ are the unitary matrices which appear into the singular value decomposition of the matrix $D_1(\lambda_0)$:

$$U^* D_1(\lambda_0)V = \operatorname{diag}(\tau_1, \ldots, \tau_{p(\lambda_0)}, \tau_{p(\lambda_0)+1}, \ldots, \tau_{n_2}). \tag{4.4}$$

And λ_0 is also a derogatory eigenvalue of the matrix G_{Y_1}; in fact, its geometric multiplicity is equal to 2.

Proof: Recall that we denoted by $\mathcal{N}$ the set of matrices $Y \in \mathbb{C}^{n_2\times n_2}$ such that the matrix G_Y is derogatory.

Let us call

$$\mathcal{C} := \{\|Y - D\| : \ Y \in \mathcal{N}\}$$

and

$$\mathcal{C}_\lambda := \{\|Y - D\| : \ Y \in \mathcal{N}_\lambda\}$$

for each $\lambda \in \Omega$. Then, by (2.7)

$$\mathcal{C} = \bigcup_{\lambda\in\Omega} \mathcal{C}_\lambda.$$

Because 0 is a lower bound of $\mathcal{C}$ and of $\mathcal{C}_\lambda$ for each $\lambda \in \Omega$, by [4, Proposition 2.3.6] we have

$$\inf \mathcal{C} = \inf\Big(\bigcup_{\lambda\in\Omega} \mathcal{C}_\lambda\Big) = \inf_{\lambda\in\Omega} \left(\inf \mathcal{C}_\lambda\right). \tag{4.5}$$

Moreover, for all $\lambda \in \Omega$

$$\inf \mathcal{C}_\lambda = \min_{Y\in\mathcal{N}_\lambda} \|Y - D\|, \tag{4.6}$$

since $\mathcal{N}_\lambda$ is a closed set (due to the lower semicontinuity of the function $X \mapsto \operatorname{rank}(X)$) .

On the other hand, by Theorem 1.1, for each λ in Ω,

$$\sigma_{p(\lambda)+1}(D_1(\lambda)) = \min_{\substack{X \in \mathbb{C}^{n_2\times n_2} \\ \operatorname{rank}\left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & X \end{array}\right) \le n-2}} \left\| \left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & \lambda I_{n_2} - D \end{array}\right) - \left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & X \end{array}\right) \right\|; \tag{4.7}$$

if $X \in \mathbb{C}^{n_2 \times n_2}$ is any matrix such that

$$\operatorname{rank}\left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & X \end{array}\right) \leq n-2$$

and we define $X' := \lambda I_{n_2} - X$, then $X' \in \mathcal{N}_\lambda$; conversely, if $X' \in \mathcal{N}_\lambda$ and $X := \lambda I_{n_2} - X'$, then

$$\operatorname{rank}\left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & X \end{array}\right) \leq n-2.$$

Consequently, for each $\lambda \in \Omega$, by virtue of (4.7),

$$\begin{aligned} \sigma_{p(\lambda)+1}(D_1(\lambda)) &= \min_{X' \in \mathcal{N}_\lambda} \left\| \left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & \lambda I_{n_2} - D \end{array}\right) - \left(\begin{array}{c|c} \lambda I_{n_1} - A & -B \\ \hline -C & \lambda I_{n_2} - X' \end{array}\right) \right\| \\ &= \min_{X' \in \mathcal{N}_\lambda} \left\| \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & (\lambda I_{n_2} - D) - (\lambda I_{n_2} - X') \end{array}\right) \right\| \\ &= \min_{X' \in \mathcal{N}_\lambda} \|X' - D\| . \end{aligned} \tag{4.8}$$

¿From (4.5), (4.6) and (4.8) we deduce

$$\begin{aligned} \min_{Y \in \mathcal{N}} \|Y - D\| &= \inf_{\lambda \in \Omega} \sigma_{p(\lambda)+1}\big(D_1(\lambda)\big) \\ &= \inf_{\lambda \in \Omega} h_{n-2}(\lambda) = \min_{\lambda \in \Omega} h_{n-2}(\lambda). \end{aligned}$$

Now let $\lambda_0 \in \Omega$ be such that

$$h_{n-2}(\lambda_0) = \min_{\lambda \in \Omega} h_{n-2}(\lambda). \tag{4.9}$$

Let $\tau_1, \ldots, \tau_{n_2}$ be the singular values of $D_1(\lambda_0)$ in nonincreasing order. By the singular value decomposition theorem, there exist unitary matrices $U, V \in \mathbb{C}^{n_2 \times n_2}$ such that

$$U^* D_1(\lambda_0) V = \operatorname{diag}(\tau_1, \ldots, \tau_{p(\lambda_0)}, \tau_{p(\lambda_0)+1}, \ldots, \tau_{n_2}).$$

By definition of h_{n-2}, see (3.8), we have

$$h_{n-2}(\lambda_0) = \sigma_{p(\lambda_0)+1}\big(D_1(\lambda_0)\big) = \tau_{p(\lambda_0)+1}. \tag{4.10}$$

Next we define

$$Y_1 := D + U \operatorname{diag}(0, \ldots, 0, \tau_{p(\lambda_0)+1}, \ldots, \tau_{n_2}) V^*. \tag{4.11}$$

As the spectral norm is unitarily invariant it follows that

$$\|Y_1 - D\| = \|\operatorname{diag}(0, \ldots, 0, \tau_{p(\lambda_0)+1}, \ldots, \tau_{n_2})\| = \tau_{p(\lambda_0)+1}. \tag{4.12}$$

Still it remains to prove that $Y_1 \in \mathcal{N}$. In fact, we are going to prove that $Y_1 \in \mathcal{N}_{\lambda_0}$. Indeed, calling $\Delta_0 := \operatorname{diag}(0, \ldots, 0, \tau_{p(\lambda_0)+1}, \ldots, \tau_{n_2})$,

$$\begin{aligned} &\operatorname{rank}\left(\begin{array}{c|c} \lambda_0 I_{n_1} - A & -B \\ \hline -C & \lambda_0 I_{n_2} - Y_1 \end{array}\right) \\ &= \operatorname{rank}\left(\begin{array}{c|c} \lambda_0 I_{n_1} - A & -B \\ \hline -C & \lambda_0 I_{n_2} - D - U \Delta_0 V^* \end{array}\right) = n-2, \end{aligned}$$

because, by (1.3), subtracting $U\Delta_0 V^*$ to the matrix $\lambda_0 I_{n_2} - D$ we attain to lower the rank of the matrix

$$\left(\begin{array}{c|c} \lambda_0 I_{n_1} - A & -B \\ \hline -C & \lambda_0 I_{n_2} - D \end{array}\right)$$

to the value

$$\operatorname{rank}\left(\begin{array}{c|c} \lambda_0 I_{n_1} - A & -B \\ \hline -C & \lambda_0 I_{n_2} - D - U\Delta_0 V^* \end{array}\right) = n-2.$$

□

Example 4.1 Now let us illustrate this theorem returning to Example 2.5. Consider again the nonderogatory matrix

$$G := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|cc} 0 & 1 & 2 \\ \hline 0 & 0 & 3 \\ 0 & 0 & 0 \end{array}\right).$$

With the same notations as at the beginning of Section 3 and the ones of last theorem, we have that if $\lambda \in \mathbb{C}$ is nonzero, then $\operatorname{rank}(\lambda I_1 - A) = 1$ and $M(\lambda) = 0, N(\lambda) = 0$; hence, $\rho_1(\lambda) = 1$. For $\lambda = 0$, we have $0I_1 - A = (0), M(0) = -B, N(0) = 0$; therefore $\rho_1(0) = \operatorname{rank} B = 1$. So, in this example, $\Omega = \mathbb{C}$. For all $\lambda \in \Omega$,

$$p(\lambda) = 3 - 2 - 1 = 0,$$

and, then, the function $h_{n-2} = h_1 : \Omega \to \mathbb{R}$ is given by

$$h_1(\lambda) = \begin{cases} \sigma_1(\lambda I_2 - D) = \|\lambda I_2 - D\| \text{ if } \lambda \neq 0, \\ \sigma_1(-D(I_2 - B^\dagger B)) \text{ if } \lambda = 0. \end{cases}$$

We see that

$$\lim_{\lambda \to 0} h_1(\lambda) = \lim_{\lambda \to 0} \sigma_1(\lambda I_2 - D) = \| - D\| = 3,$$

but $h_1(0) = 3/\sqrt{5}$. This implies that h_1 has a removable discontinuity at 0 as it was announced in the theorem (0 is the eigenvalue of A). As for all $\lambda \neq 0$, we have

$$\left\| \begin{pmatrix} \lambda & -3 \\ 0 & \lambda \end{pmatrix} \right\| = \sqrt{\frac{2 \mid \lambda \mid^2 + 9 + \sqrt{36 \mid \lambda \mid^2 + 81}}{2}},$$

then

$$\inf_{\lambda \in \mathbb{C} \setminus \{0\}} h_1(\lambda) = 3.$$

Hence,

$$\min_{\lambda \in \Omega} h_1(\lambda) = 3/\sqrt{5}.$$

In consequence the nearest derogatory matrix of shape $\left(\begin{array}{c|c} A & B \\ \hline C & Y \end{array}\right)$ is at distance $3/\sqrt{5}$ from $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$. Using the singular value decomposition of

$$D_1(0) = \begin{pmatrix} 1.2 & -0.6 \\ 0 & 0 \end{pmatrix},$$

we obtain

$$U^*D_1(0)V = \begin{pmatrix} 3/\sqrt{5} & 0 \\ 0 & 0 \end{pmatrix}.$$

¿From here, a 2×2 matrix Y where the minimum on the left-hand side of (4.2) is attained, comes given by

$$Y_1 := D + U\mathrm{diag}(3/\sqrt{5}, 0)V^* = \begin{pmatrix} 1.2 & 2.4 \\ 0 & 0 \end{pmatrix}.$$

In fact, the Jordan form of

$$G_{Y_1} = \left(\begin{array}{c|c} A & B \\ \hline C & Y_1 \end{array}\right) = \left(\begin{array}{c|cc} 0 & 1 & 2 \\ \hline 0 & 1.2 & 2.4 \\ 0 & 0 & 0 \end{array}\right)$$

is

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1.2 \end{pmatrix}.$$

Hence, $\mathrm{gm}(0, G_{Y_1}) = 2$, and G_{Y_1} is a derogatory matrix; in addition to this,

$$\|G_{Y_1} - G\| = 3/\sqrt{5}.$$

This corroborates what was straightforward obtained in Example 2.5.

5 Derogatory pseudospectrum

Let $M \in \mathbb{C}^{n\times n}$; we will denote by $S_2(M)$ the set of derogatory eigenvalues of M. Let $G \in \mathbb{C}^{n\times n}$ be the nonderogatory partitioned matrix

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with $A \in \mathbb{C}^{n_1\times n_1}, B \in \mathbb{C}^{n_1\times n_2}, C \in \mathbb{C}^{n_2\times n_1}$ and $D \in \mathbb{C}^{n_2\times n_2}$.

Where are the derogatory eigenvalues of all matrices

$$G_Y := \begin{pmatrix} A & B \\ C & Y \end{pmatrix}$$

such that $Y \in \mathbb{C}^{n_2\times n_2}$, is sufficiently close to D? This question is closely related with the problem treated in Section 4. We would like to find out the geometric description of the set in the complex plane formed by the derogatory eigenvalues of all the matrices G_Y whose distance from G is less than or equal to a prefixed $\varepsilon > 0$. If ε is less than

$$\min_{\lambda\in\Omega} h_{n-2}(\lambda),$$

then there is no derogatory eigenvalue of the matrices G_Y where $\|Y - D\| \leq \varepsilon$, because, by (4.2), all these matrices are nonderogatory. So, a necessary condition for the set

$$\bigcup_{\|Y-D\|\leq\varepsilon} S_2(G_Y) \quad \text{(derogatory pseudospectrum of } G \text{ of radius } \varepsilon)$$

be nonempty is that

$$\varepsilon \geq \min_{\lambda \in \Omega} h_{n-2}(\lambda).$$

It is natural that the derogatory pseudospectrum of G of radius ε is equal to the set enclosed by the ε–level curve of the function $f(x,y) := h_{n-2}(x+yi)$. This fact is consequence of the following theorem.

Theorem 5.1 *With the preceding notations, let $\varepsilon > 0$ be a real number. Then*

$$\bigcup_{\substack{Y \in \mathbb{C}^{n_2 \times n_2} \\ \|Y-D\| \leq \varepsilon}} S_2(G_Y) = \{z \in \Omega : h_{n-2}(z) \leq \varepsilon\}. \tag{5.1}$$

Proof: Recall that

$$\Omega = \{\lambda \in \mathbb{C} : \rho_1(\lambda) \leq n-2\} = \{\lambda \in \mathbb{C} : \mathcal{N}_\lambda \neq \emptyset\}.$$

Let $z \in \Omega$ be such that $h_{n-2}(z) \leq \varepsilon$; then

$$\sigma_{p(z)+1}\big(D_1(z)\big) \leq \varepsilon. \tag{5.2}$$

But

$$\sigma_{p(z)+1}\big(D_1(z)\big) =$$

$$\min_{\substack{X \in \mathbb{C}^{n_2 \times n_2} \\ \operatorname{rank}\left(\begin{array}{c|c} zI_{n_1}-A & -B \\ \hline -C & X \end{array}\right) \leq n-2}} \left\| \left(\begin{array}{c|c} zI_{n_1}-A & -B \\ \hline -C & zI_{n_2}-D \end{array}\right) - \left(\begin{array}{c|c} zI_{n_1}-A & -B \\ \hline -C & X \end{array}\right) \right\|;$$

$$= \min_{X' \in \mathcal{N}_z} \|X' - D\| = \|X'_0 - D\|,$$

with $X'_0 \in \mathcal{N}_z$ (what implies z is a derogatory eigenvalue of $G_{X'_0}$).

Furthermore, from (5.2) we have $\|X'_0 - D\| \leq \varepsilon$. Hence

$$\{z \in \Omega : h_{n-2}(z) \leq \varepsilon\} \subset \bigcup_{\|Y-D\| \leq \varepsilon} S_2(G_Y).$$

Reciprocally, if $z \in S_2(G_{X'_0})$ for some $X'_0 \in \mathbb{C}^{n_2 \times n_2}$ such that $\|X'_0 - D\| \leq \varepsilon$, it follows that $X'_0 \in \mathcal{N}_z$; this implies $\mathcal{N}_z \neq \emptyset$, so $z \in \Omega$. Besides,

$$\|X'_0 - D\| = \left\| \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & (zI_{n_2}-D) - (zI_{n_2}-X'_0) \end{array}\right) \right\| =$$

$$\left\| \left(\begin{array}{c|c} zI_{n_1}-A & -B \\ \hline -C & zI_{n_2}-D \end{array}\right) - \left(\begin{array}{c|c} zI_{n_1}-A & -B \\ \hline -C & zI_{n_2}-X'_0 \end{array}\right) \right\| \geq$$

$$\sigma_{p(z)+1}\big(D_1(z)\big) = h_{n-2}(z);$$

therefore it implies $\varepsilon \geq h_{n-2}(z)$. Hence

$$\bigcup_{\|Y-D\| \leq \varepsilon} S_2(G_Y) = \{z \in \Omega : h_{n-2}(z) \leq \varepsilon\}.$$

□

Example 5.1 Now we consider again the nonderogatory matrix

$$G := \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 1 & 2 & 3 \\ 0 & 2 & 4 & 5 \\ \hline 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

of Example 3.1. Here $n - 2 = 4 - 2 = 2$, $\Omega = \mathbb{C}(\equiv \mathbb{R}^2)$ and the function h_{n-2} is

$$f(x, y) := h_2(x + yi),$$

defined in all $\mathbb{R}^2$. The graphic of the surface $z = f(x, y)$ is shown in figure 1. Moreover, as we say there

$$\min_{(x,y)\in\mathbb{R}^2} f(x, y) = f(2, 0) \approx 3.2196$$

In addition to this, $f(1, 0) \approx 3.5355$, the minimum of f on $\Omega \setminus \{(1, 0), (2, 0)\}$ exists and

$$\min_{(x,y)\in\Omega\setminus\{(1,0),(2,0)\}} f(x, y) \approx 6.0414;$$

what is the meaning of this value? If $\varepsilon < 3.2196$ then the derogatory pseudospectrum of G of radius ε is empty, because for all $Y \in \mathbb{C}^{2\times 2}$ such that $\|Y - D\| \leq \varepsilon$, the matrix G_Y is nonderogatory. If $3.2196 \leq \varepsilon < 3.5355$ then the derogatory pseudospectrum of G of radius ε consists of $\{1\}$. In the case of $3.5355 \leq \varepsilon < 6.0414$, then the derogatory pseudospectrum of G of radius ε consists of $\{1, 2\}$. Finally, if $\varepsilon > 6.0414$, then the derogatory pseudospectrum has positive area. Figure 3 shows these derogatory pseudospectra, the sublevel region $\{(x, y) \in \mathbb{R}^2 : f(x, y) \leq \varepsilon\}$, for $\varepsilon = 6.060, 6.395, 6.946, 8.000$.

Restricted pseudospectrum

Let $G \in \mathbb{C}^{n\times n}$ be the partitioned matrix

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with $A \in \mathbb{C}^{n_1\times n_1}, B \in \mathbb{C}^{n_1\times n_2}, C \in \mathbb{C}^{n_2\times n_1}$ and $D \in \mathbb{C}^{n_2\times n_2}$. We suppose $n_1 \geq 1, n_2 \geq 1$. Here G is any matrix, not necessarily nonderogatory.

Where are the eigenvalues of all matrices

$$G_Y := \begin{pmatrix} A & B \\ C & Y \end{pmatrix}$$

such that $Y \in \mathbb{C}^{n_2\times n_2}$ is sufficiently close to D? This question is closely related with the problem treated in the first part of this Section 5. We would like to find out the geometric description of the set in the complex plane formed by the eigenvalues of all the matrices G_Y whose distance from G is less than or equal to a prefixed $\varepsilon > 0$. The same question, if it is permitted to perturb

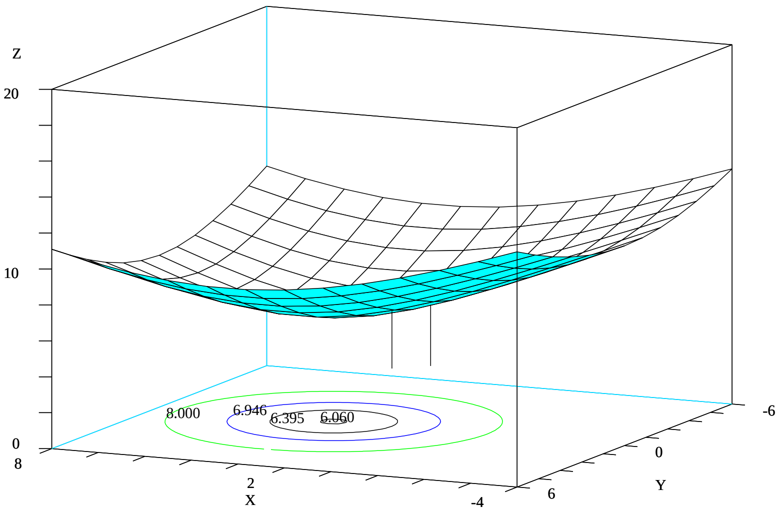

Figure 3: Derogatory pseudospectra of G

in all entries of the matrix G, has been studied in [10], [11], with the name of pseudospectrum of the matrix G of radius $\varepsilon > 0$:

$$\bigcup_{\substack{G' \in \mathbb{C}^{n\times n} \\ \|G'-G\| \leq \varepsilon}} \sigma(G').$$

It was proved that

$$\bigcup_{\substack{G' \in \mathbb{C}^{n\times n} \\ \|G'-G\| \leq \varepsilon}} \sigma(G') = \{z \in \mathbb{C} : \sigma_n(zI_n - G) \leq \varepsilon\},$$

where $\sigma_n(zI_n - G)$ is the minimum singular value of the matrix $zI_n - G$.

For every $\lambda \in \mathbb{C}$, define

$$\mathcal{N}_{1,\lambda} := \left\{ Y \in \mathbb{C}^{n_2 \times n_2} : \lambda \text{ is an eigenvalue of } \begin{pmatrix} A & B \\ C & Y \end{pmatrix} \right\}.$$

and let Ω_1 be the set $\{\lambda \in \mathbb{C} : \mathcal{N}_{1,\lambda} \neq \emptyset\}$. Given the matrices A, B and C, can it happen that for some $\lambda \in \mathbb{C}$ the set $\mathcal{N}_{1,\lambda}$ be empty? The answer is affirmative as we can see that for all $y \in \mathbb{C}$

$$\det \begin{pmatrix} \lambda - 1 & -2 \\ -3 & \lambda - y \end{pmatrix} = (\lambda - 1)(\lambda - y) - 6;$$

So if $\lambda = 1$, there is no y such that 1 be an eigenvalue of

$$\left(\begin{array}{c|c} 1 & 2 \\ \hline 3 & y \end{array} \right).$$

In fact, in this example $\Omega_1 = \mathbb{C} \setminus \{1\}$. Calling for any $\lambda \in \Omega_1$, $\rho_1(\lambda)$ as in (3.5) and $D_1(\lambda)$ as in (3.7), it is simple to see that $\Omega_1 = \{\lambda \in \mathbb{C} : \rho_1(\lambda) \leq n-1\}$. We have always $\mathbb{C} \setminus \sigma(A) \subset \Omega_1$, because for all $\lambda \in \mathbb{C} \setminus \sigma(A)$ it follows $\rho_1(\lambda) = n_1$ and $n_1 \leq n-1$.

Now we define the set $\Omega^{(1)} := \{\lambda \in \mathbb{C} : n_1 \leq \rho_1(\lambda) \leq n-1\}$. Obviously $\Omega^{(1)} \subset \Omega_1$, but the content can be strict. For example, given the matrix

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left(\begin{array}{ccc|cc} 3 & 1 & -2 & 0 & 0 \\ 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ \hline 0 & 0 & 0 & 6 & -4 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right),$$

we have that $3 \in \Omega_1$, because 3 is an eigenvalue of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$; but $\rho_1(3) = 2 \not\geq 3$, so $3 \notin \Omega^{(1)}$.

For all $\lambda \in \Omega^{(1)}$, we define $p_1(\lambda) := n - 1 - \rho_1(\lambda)$ and the function

$$h_{n-1} : \Omega^{(1)} \to \mathbb{R}$$

by $h_{n-1}(\lambda) := \sigma_{p_1(\lambda)+1}\big(D_1(\lambda)\big)$. It is easy to see that this has meaning given that for all $\lambda \in \Omega^{(1)}$, $0 \leq p_1(\lambda) \leq n_2 - 1$. Moreover, $\lambda \in \sigma(G) \cap \Omega^{(1)}$ if and only if $h_{n-1}(\lambda) = 0$. By an analogous way of the proof of Theorem 5.1 we can prove the following result.

Theorem 5.2 *Let $G \in \mathbb{C}^{n\times n}$ be the partitioned matrix*

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with $A \in \mathbb{C}^{n_1\times n_1}, B \in \mathbb{C}^{n_1\times n_2}, C \in \mathbb{C}^{n_2\times n_1}$ and $D \in \mathbb{C}^{n_2\times n_2}$. And let $\varepsilon > 0$ be a real number. Then

$$\bigcup_{\substack{Y\in\mathbb{C}^{n_2\times n_2}\\ \|Y-D\|\leq\varepsilon}} \sigma(G_Y) = \{z \in \Omega^{(1)} : h_{n-1}(z) \leq \varepsilon\} \cup \sigma(G).$$

There is an alternative characterization of the restricted pseudospectrum of G of radius $\varepsilon > 0$

$$\bigcup_{\substack{Y\in\mathbb{C}^{n_2\times n_2}\\ \|Y-D\|\leq\varepsilon}} \sigma(G_Y),$$

as

$$\bigcup_{\substack{Y\in\mathbb{C}^{n_2\times n_2}\\ \|Y-D\|\leq\varepsilon}} \sigma(G_Y) = \big\{z \in \mathbb{C} \setminus \sigma(G) : \sigma_{n_2}\big(R(z)\big) \leq \varepsilon\big\} \cup \sigma(G),$$

where

$$R(z) := \left[(0, I_{n_2})\,(zI_n - G)^{-1} \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix}\right]^{\dagger}$$

is the Moore-Penrose inverse of a transfer matrix. See [7, Proposition 2.3, p. 128].

Conclusions

In [12] it was reformulated a result of [3] that gives in a precise manner how to find the nearest matrix $\left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right)$ that lowers the rank of the matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, by means of ordinary singular values of a matrix related with A, B, C and D through the Moore-Penrose inverse. Given that many important features of the Jordan canonical form of a matrix (in particular, the geometric multiplicity of its eigenvalues) can be formulated in terms of ranks of certain matrices, we have been able to obtain a solution to related nearness matrix problems from this theorem.

We have obtained the nearest derogatory matrix $\left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right)$ to the nonderogatory matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ if we perturb only in D. Also, we have established the relation of this last problem with the question of where are the derogatory eigenvalues of matrices $\left(\begin{smallmatrix} A & B \\ C & Y \end{smallmatrix}\right)$ with Y adequately close to D.

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